ANALYTIC TORSION AND RUELLE ZETA FUNCTIONS FOR HYPERBOLIC MANIFOLDS WITH CUSPS

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Abstract. In this paper we derive a relationship of the leading coefficient of the Laurent expansion of the Ruelle zeta function at $s = 0$ and the analytic torsion for hyperbolic manifolds with cusps. Here, the analytic torsion is defined by a certain regularized trace following Melrose [16]. This extends the result of Fried, which was proved for the compact case in [6], to a noncompact case.
Keywords: Analytic torsion, Ruelle zeta function, Selberg trace formula.

1. Introduction

In this paper we derive an equality between the leading coefficient of the Laurent expansion of the Ruelle zeta function at $s = 0$ and the analytic torsion for hyperbolic manifolds with cusps. This extends the result of Fried, which was proved for the compact case in [6], to a noncompact case. Here the analytic torsion for manifolds with cusps is defined by a certain regularized trace following the idea of the b-trace of Melrose [16].

This paper can be considered as a continuation of our previous study [21] of the relationship between a special value of the odd type Selberg zeta function and the eta invariant which extends the result of Millson [18] to hyperbolic manifolds with cusps.

Let X_Γ denote a d-dimensional hyperbolic manifold with cusps, which is given by

$$X_\Gamma = \Gamma\backslash \mathrm{SO}_0(d,1)/\mathrm{SO}(d)$$

where Γ is a co-finite discrete subgroup in $\mathrm{SO}_0(d,1)$. We assume that X_Γ is equipped with the constant negative curvature -1. We also assume that Γ is neat (hence torsion free), that is, the group generated by the eigenvalues of Γ contains no root of unity. A consequence of this is

$$\Gamma_P := \Gamma \cap P = \Gamma \cap N(P) \qquad \text{for} \quad P \in \mathfrak{P}_\Gamma \tag{1.1}$$

where $N(P)$ is the nilpotent part of P and $\mathfrak{P}_\Gamma = \{P_1, \ldots, P_\kappa\}$ denotes a complete set of Γ-conjugacy classes of Γ-cuspidal parabolic subgroups of $\mathrm{SO}_0(d,1)$.

The Ruelle zeta function $R_\chi(s)$ over X_Γ is now defined by

$$R_\chi(s) := \prod_{\gamma \in \mathrm{P}\Gamma_{\mathrm{hyp}}} \det\Big(\mathrm{Id} - \chi(\gamma) e^{-s\, l(C_\gamma)}\Big)^{-1}$$

for $\mathrm{Re}(s) > (d-1)$. Here $\mathrm{P}\Gamma_{\mathrm{hyp}}$ denotes the set of Γ-conjugacy classes of the primitive hyperbolic elements in Γ, the determinant denoted by "det" is taken over the representation space V_χ of a unitary representation χ of Γ, and $l(C_\gamma)$ denotes the length of the closed geodesic determined by a hyperbolic element γ.

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Let us recall some results about $R_\chi(s)$ in [10] when $d = 2n+1$. First, by Theorem 1.1 of [10], the Ruelle zeta function $R_\chi(s)$ has a meromorphic extension to $\mathbb{C}$. Second, let N_0 be the order of the singularity of $R_\chi(s)$ at $s = 0$ such that

$$R^*_\chi(0) := \lim_{s\to 0} s^{N_0} R_\chi(s) \ \in\ \overline{\mathbb{C}} - \{0, \infty\}.$$

By Theorem 1.2 of [10], we have that if $d = 2n+1$,

$$N_0 \ =\ 2\sum_{k=0}^{n}(-1)^k(n+1-k)\beta_k + \sum_{k=0}^{n-1}(-1)^{k+1}b_k\binom{2n}{k} + d_c(\chi)\,(-1)^n\binom{2n-2}{n-1} \tag{1.2}$$

where $\beta_k := \dim\ker_{L^2}(\Delta_k)$ with the Hodge Laplacian Δ_k acting on the space of differential k-forms twisted by χ over X_Γ, b_k denotes the order of the singularity of the determinant of a certain scattering operator $C^k_\chi(\sigma_k, s)$ at $s = \frac{d-1}{2} - k$, and $d_c(\chi)$ is the sum of the dimensions of the maximal subspaces of V_χ's where $\chi|_{\Gamma\cap P}$ acts trivially for $P \in \mathfrak{P}_\Gamma$ (see (3.2)).

From (1.2), we can see that if $d = 2n+1$ the behavior of the Ruelle zeta function $R_\chi(s)$ at $s = 0$ is related to the spectral data of the Hodge Laplacians Δ_k's, and it is a natural question whether the leading coefficient $R^*_\chi(0)$ may have a relationship with another spectral data. In [6], it was proved that this is equal to the analytic torsion (up to a constant) for odd-dimensional compact hyperbolic manifold X_Γ. Since we do not have an analytic torsion for our noncompact case, we need to introduce an analytic torsion $T(X_\Gamma, \chi)$ which is linked to the leading coefficient $R^*_\chi(0)$. To do this, first of all we define the spectral zeta function of the Hodge Laplacian Δ_k using a certain regularized trace of the heat operator of Δ_k. In Theorem 6.1 we show that this spectral zeta function of Δ_k is regular at $s = 0$. Then we can define the regularized determinants of Δ_k's and the analytic torsion $T(X_\Gamma, \chi)$ in the usual way as in the compact case. Actually this approach was suggested by Melrose in [16] and the regularized trace in this paper is essentially the same as the b-trace of Melrose. Following this idea, Hassell defined b-analytic torsion for certain noncompact manifolds in [12]. We refer to Section 6 for the precise definitions of the spectral zeta function of Δ_k and $T(X_\Gamma, \chi)$.

The following theorem states a relationship of $R^*_\chi(0)$ with the analytic torsion $T(X_\Gamma, \chi)$ where some defect terms are given from the cusps (geometrically) and the scattering data of Δ_k's (analytically).

Theorem 1.1. *For a $(2n+1)$-dimensional hyperbolic manifold X_Γ with cusps, the following equality holds up to sign,*

$$R^*_\chi(0)^{-1} = C(X_\Gamma, \chi)\cdot C(d)^{d_c(\chi)}\cdot S(X_\Gamma, \chi)\cdot T(X_\Gamma, \chi). \tag{1.3}$$

Here

$$C(X_\Gamma, \chi) := \prod_{k=0}^{n-1}\left(-4(n-k)^2\right)^{(-1)^k\alpha_k}, \qquad C(d) := \prod_{k=0}^{n-1} 2^{(-1)^{k+1}e_1}\cdot(n-k)^{(-1)^k e_2}$$

where $\alpha_k := \beta_k - \beta_{k-1} + \beta_{k-2} - \ldots \pm \beta_0$, $e_1 = \binom{2n}{k} - \binom{2n-1}{k}$, $e_2 = (2n-2k+1)\binom{2n}{k} - \binom{2n-1}{k}$,

$$S(X_\Gamma, \chi) := \prod_{k=0}^{n-1} S_\chi(k)^{(-1)^{k+1}\binom{2n}{k}} \quad \textit{with} \quad S_\chi(k) = \lim_{s\to-(n-k)}(s+n-k)^{-b_k}\det C^k_\chi(\sigma_k, s).$$

This result was announced in [22].

Remark 1.2. Let us observe that $C(d)$ depends only on the dimension d, not on Γ although $C(X_\Gamma, \chi)$, $S(X_\Gamma, \chi)$ depend on Γ sensitively. When X_Γ is compact, the equality (1.3) is reduced to the formula of Fried in [6]. Actually we can see that the same formula holds under a more general condition that $d_c(\chi) = 0$ even if X_Γ may have cusps. In fact, if $d_c(\chi) = 0$, then N_0 is given only by the β_k's in (1.2) and $C(d)^{d_c(\chi)} = S(X_\Gamma, \chi) = 1$. Moreover the sign ambiguity in Theorem 1.1 disappears since this comes from the scattering operators $C^k_\chi(\sigma_k, s)$.

Remark 1.3. In [26], [27], Sugiyama studied the geometric analogues of the Iwasawa conjecture for 3-dimensional hyperbolic manifolds. He proved that the Laurent expansion of the Ruelle zeta function $R_\chi(s)$ at $s = 0$ satisfies several analogues of the Iwasawa conjecture in the algebraic number theory under the condition $d_c(\chi) = 0$. In particular, in [27] it is proved that $R^*_\chi(0)$ is essentially given by the Reidemeister torsion for (X_Γ, χ) if $d_c(\chi) = 0$. Our Theorem 1.1 is crucially used in its proof. It seems to be interesting to understand the equality (1.3) in Theorem 1.1 for general cases in the view point of the geometric analogues of the Iwasawa conjecture.

Comparing the formulae of the order of the singularity N_0 for even- and odd-dimensional cases (see (1.2) and Theorem 1.2 of [10]), one can expect that there is less relationship of $R^*_\chi(0)$ with the spectral data in the even-dimensional case. Actually because of a certain symmetry (see (8.6)) we can not link $R^*_\chi(0)$ with $T(X_\Gamma, \chi)$ in the even-dimensional case. It is also known that the analytic torsion $T(X_\Gamma, \chi)$ is trivial for even-dimensional compact manifold. However, it turned out that this is not true anymore for noncompact hyperbolic manifold X_Γ with cusps, and that $T(X_\Gamma, \chi)$ has an following explicit expression.

Theorem 1.4. *For a $2n$-dimensional hyperbolic manifold X_Γ with cusps, the following equality holds*

$$T(X_\Gamma, \chi) = \Big(\prod_{k=0}^{n-1} (n - 1/2 - k)^{(-1)^k n \left(\binom{2n-1}{k+1} - \binom{2n-2}{k} \right)} \Big)^{d_c(\chi)}. \tag{1.4}$$

Remark 1.5. The right-hand side of (1.4) originates from the non-invariant part of the weighted unipotent orbital integral on the geometric side of the Selberg trace formula. Geometrically this is the defect for the Hodge theorem of the de Rham complex for hyperbolic manifolds with cusps. For the odd-dimensional case, the corresponding term is also contained in the factor $C(d)$ in Theorem 1.1.

Remark 1.6. For the case of $d = 2n$, we can also obtain an expression of $R^*_\chi(0)$ in terms of similar factors on the right-hand side of (1.3) except $T(X_\Gamma, \chi)$. This easily follows from the functional equation of $R_\chi(s)$ presented in Theorem 1.1 of [10]. (The simplest case of $d = 2$ was also mentioned at pp. 162 in [5].)

Now let us explain the structure of this paper. In Section 2, we review the basics of harmonic analysis over hyperbolic spaces to fix notations and normalizations used in this paper. In Section 3 we study the spectral side of the Selberg trace formula. This will explain the motivation of the regularized trace that is used to define the analytic torsion for hyperbolic manifolds with cusps. In Section 4 we explain the Selberg trace formula for the nontrivial homogeneous vector bundles over hyperbolic manifolds with cusps. In Section 5, we completely compute the contribution of the weighted orbital integrals for our case applying the result in [13]. In Section 6, we define the spectral zeta functions of the Hodge Laplacians using the regularized trace following Melrose [16] and show that they have meromorphic extensions over $\mathbb{C}$. This enables us to define the regularized determinant and analytic torsion. In Section 7 and 8, we prove Theorem 1.1 and 1.4 combining all the results proved in the previous sections. In Appendix A, we perform an algebraic computation which gives the proof of Theorem 5.3.

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2. Harmonic analysis over real hyperbolic space

2.1. **Algebraic structures.** The d-dimensional real hyperbolic space is the manifold

$$\mathrm{H}^d(\mathbb{R}) = \left\{ x \in \mathbb{R}^{d+1} \,|\, x_1^2 + x_2^2 + \ldots + x_d^2 - x_{d+1}^2 = -1,\ x_{d+1} > 0 \right\}$$

equipped with the metric of curvature -1. The orientation preserving isometries of $\mathrm{H}^d(\mathbb{R})$ form the group $G = \mathrm{SO}_0(d,1)$ which is the identity component of $\mathrm{SO}(d,1)$. The isotropy subgroup K of the base point $(0,\ldots,0,1)$ is isomorphic to $\mathrm{SO}(d)$. Hence the real hyperbolic space $\mathrm{H}^d(\mathbb{R})$ can be identified with the symmetric space G/K. We denote the Lie algebras of G, K by $\mathfrak{g} = \mathfrak{so}(d,1)$, $\mathfrak{k} \cong \mathfrak{so}(d)$ respectively. The Cartan involution θ on $\mathfrak{g}$ gives us the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}, \mathfrak{p}$ are the $1, -1$ eigenspaces of θ, respectively. The subspace $\mathfrak{p}$ can be identified with the tangent space $T_o(G/K) \cong \mathfrak{g}/\mathfrak{k}$ at $o = eK \in G/K$. The invariant metric of curvature -1 over $\mathrm{H}^d(\mathbb{R})$ corresponds to the normalized Cartan-Killing form

$$\langle X, Y\rangle := -\frac{1}{2(d-1)} C(X, \theta Y) \tag{2.1}$$

where the Killing form is defined by $C(X,Y) = \mathrm{Tr}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y)$ for $X, Y \in \mathfrak{g}$.

Let $\mathfrak{a}$ be a fixed maximal abelian subspace of $\mathfrak{p}$. Then the dimension of $\mathfrak{a}$ is one. Let $M \cong \mathrm{SO}(d-1)$ be the centralizer of $A = \exp(\mathfrak{a})$ in K with Lie algebra $\mathfrak{m}$. Let T_M be a Cartan subgroup in M so that $T = T_M \cdot A$ is a Cartan subgroup of G. Let Σ_M be the half system of the positive roots for $(\mathfrak{m}_\mathbb{C}, \mathfrak{t}_{\mathfrak{m}\mathbb{C}})$. We choose the half system Σ_A of positive roots of $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ which do not vanish on $\mathfrak{a}_\mathbb{C}$ so that Σ_A is compatible with Σ_M. Then the union of Σ_M with Σ_A gives the half system of positive roots for $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, which is denoted by Σ_G. With respect to the inner product on $\mathfrak{t}_\mathbb{C}^*$ induced from $\langle \cdot, \cdot \rangle$ in (2.1), we choose an orthonormal basis $\{e_i\}$ of $\mathfrak{t}_\mathbb{C}^*$ such that $e_1 \in \mathfrak{a}_\mathbb{C}^*$. Then we have

(1) When $d = 2n+1$,

$$\begin{aligned}\Sigma_G &= \{\ e_i + e_j\ (1 \le i < j \le n+1), \quad e_i - e_j\ (1 \le i < j \le n+1)\ \},\\ \Sigma_A &= \{\ e_1 + e_j\ (1 < j \le n+1), \quad e_1 - e_j\ (1 < j \le n+1)\ \}.\end{aligned}$$

(2) When $d = 2n$,

$$\begin{aligned}\Sigma_G &= \{\ e_i\ (1 \le i \le n), \quad e_i - e_j\ (1 \le i < j \le n), \quad e_i + e_j\ (1 \le i < j \le n)\ \},\\ \Sigma_A &= \{\ e_1, \quad e_1 - e_j\ (1 < j \le n), \quad e_1 + e_j\ (1 < j \le n)\ \}.\end{aligned}$$

We put $\beta = e_1$, which is the positive restricted root of $(\mathfrak{g}, \mathfrak{a})$. Let ρ denote the half sum of the positive roots of $(\mathfrak{g}, \mathfrak{a})$, that is, $\rho = \frac{(d-1)}{2}\beta$. Later on, we shall use the identification

$$\mathfrak{a}_\mathbb{C}^* \cong \mathbb{C} \qquad \text{by} \quad \lambda\beta \longrightarrow \lambda. \tag{2.2}$$

Let $\mathfrak{n}$ be the positive root space of β and $N = \exp(\mathfrak{n}) \subset G$. The Iwasawa decomposition is given by $G = KAN$. From now on we fix the following Haar measure on G,

$$dg = a^{2\rho} dk\, da\, dn = a^{-2\rho} dn\, da\, dk \tag{2.3}$$

where $g = kan$ is the Iwasawa decomposition and $a^{2\rho} = \exp(2\rho(\log a))$. Here dk is the Haar measure over K with $\int_K dk = 1$, da is the Euclidean Lebesgue measure on A given by the identification $A \cong \mathbb{R}$ via $a_t = \exp(tH)$ with $H \in \mathfrak{a}$, $\beta(H) = 1$, and dn is the Euclidean Lebesgue measure on N induced by the normalized Cartan-Killing form $\langle \cdot, \cdot \rangle$ given in (2.1).

2.2. **Homogeneous vector bundle.** Let us recall the homogeneous vector bundle over the symmetric space $\mathrm{H}^d(\mathbb{R}) \cong G/K$. If τ is a unitary finite-dimensional representation of K, then the sections of the associated homogeneous vector bundle $G \times_\tau V_\tau$ over G/K consist of the map $f : G \to V_\tau$ with the condition

$$f(gk) = \tau(k)^{-1} f(g) \qquad \text{for} \quad g \in G, k \in K. \tag{2.4}$$

Equivalently, the sections of $G \times_\tau V_\tau$ are equivalence classes of the pairs (g, v) under $(gk, v) \sim (g, \tau(k)v)$. For such a section f of $G \times_\tau V_\tau$, there is a G-action defined by $g_0 \cdot f(g) = g_0 f(g_0^{-1} g)$. For instance, the space of k-forms over G/K is given by this construction: Choose an orthonormal basis for $\mathfrak{p}^*$. This basis determines left invariant 1-forms $\omega_1, \ldots, \omega_d$ on G. A complex valued k-form w on G/K pulls back to a k-form ω' on G given by

$$\omega' = \sum f_{i_1,\ldots,i_k} \omega_{i_1} \wedge \ldots \omega_{i_k}.$$

The component functions $(f_{i_1,\ldots,i_k})$ give a map $f : G \to \wedge^k \mathbb{C}^d$ satisfying the condition (2.4) with $\tau = \tau_k$ acting on $V_{\tau_k} \cong \wedge^k \mathbb{C}^d$. All the representations τ_k are irreducible representations of K except when $d = 2n$ and $k = n$. In this case, τ_n decomposes into two irreducible representations τ_n^+, τ_n^- acting on $\wedge^n_+ \mathbb{C}^{2n}, \wedge^n_- \mathbb{C}^{2n}$, respectively. Here $\wedge^n_\pm \mathbb{C}^{2n}$ denotes the $\pm \exp(\frac{n^2}{2}\pi i)$-eigenspace of the Hodge operator $*$ on $\wedge^n \mathbb{C}^{2n}$. Let us recall that the highest weight μ_k of the representation τ_k is given by

$$\begin{aligned}
\mu_k &= e_2 + e_3 + \ldots + e_{k+1} && \text{for} \quad 1 \le k \le n, && d = 2n+1, \\
\mu_k &= e_2 + e_3 + \ldots + e_{k+1} && \text{for} \quad 1 \le k \le n-1, && d = 2n, \\
\mu_n^\pm &= e_2 + e_3 + \ldots + e_n \pm e_{n+1} && \text{for} \quad d = 2n. &&
\end{aligned}$$

Let us denote the irreducible fundamental representations of $M = \mathrm{SO}(d-1)$ by σ_k's if $d = 2n$, and σ_k's with $k \neq n$, $\sigma_n^\pm$ if $d = 2n+1$. These satisfy the following branching laws:

(1) For $k \neq n$ with $d = 2n$ or $d = 2n+1$,

$$[\tau_k|_M : \sigma_\ell] = 1 \qquad \text{if and only if} \qquad \sigma_\ell = \sigma_k \quad \text{or} \quad \sigma_\ell = \sigma_{k-1},$$

(2) For $k = n$ and $d = 2n$,

$$[\tau_n^\pm|_M : \sigma_\ell] = 1 \qquad \text{if and only if} \qquad \sigma_\ell = \sigma_n = \sigma_{n-1},$$

(3) For $k = n$ and $d = 2n+1$,

$$[\tau_n|_M : \sigma_\ell] = 1 \qquad \text{if and only if} \qquad \sigma_\ell = \sigma_{n-1} \quad \text{or} \quad \sigma_\ell = \sigma_n^\pm.$$

2.3. **Heat kernel of the Hodge Laplacian.** The Hodge Laplacian Δ_k on the space of k-forms for the curvature -1 metric is an invariant differential operator constructed as follows: We choose a basis E_i of $\mathfrak{k}$ and a basis E_j of $\mathfrak{p}$ that are orthonormal with respect to the normalized Cartan-Killing form $\langle \cdot, \cdot \rangle$. Then

$$\Omega := -\sum E_i^2 + \sum E_j^2 \tag{2.5}$$

is the normalized Casimir element in the center of the universal enveloping algebra of $\mathfrak{g}$. For a representation τ of K, let

$$Q_\tau = \int_K R(k) \otimes \tau(k)\, dk$$

be the projection from $L^2(G) \otimes V_\tau$ to $L^2(G, \tau) \cong (L^2(G) \otimes V_\tau)^K$ where R denotes the right regular representation of G on $L^2(G)$. Then the Hodge Laplacian Δ_k is given by

$$\Delta_k = Q_{\tau_k} \left(R(-\Omega) \otimes \mathrm{Id}_{V_{\tau_k}} \right) Q_{\tau_k}. \tag{2.6}$$

That is, the Hodge Laplacian Δ_k is the restriction to the τ_k-invariant part of the corresponding invariant differential operator to $-\Omega$.

The subgroup $P_0 := NAM$ is a minimal parabolic subgroup of G. Given $(\sigma, H_\sigma) \in \widehat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, the following action

$$(1 \otimes e^{i\lambda} \otimes \sigma)(nam) = a^{i\lambda} \sigma(m)$$

defines a representation of P_0 on H_σ where $a^{i\lambda}$ denotes $\exp(i\lambda(\log a))$. Then the *principal series representation* $\pi_{\sigma,\lambda} := \mathrm{Ind}_{P_0}^G (1 \otimes e^{i\lambda} \otimes \sigma)$ of G acts on the space

$$\mathcal{H}_{\sigma,\lambda} := \{ f : G \to H_\sigma \mid f(namx) = a^{(i\lambda+\rho)} \sigma(m) f(x),\ f|_K \in L^2(K) \}$$

by the right translation $\pi_{\sigma,\lambda}(g) f(x) = f(xg)$. The following proposition whose proof is similar to Lemma 1 of [6] gives the action of Δ_k over $\mathcal{H}_{\sigma_\ell,\lambda}$ if $[\tau_k|_M : \sigma_\ell] \neq 0$.

Proposition 2.1. *If $[\tau_k|_M : \sigma_\ell] \neq 0$, the Hodge Laplacian Δ_k acts on $\mathcal{H}_{\sigma_\ell,\lambda}$ by $\lambda^2 + (\frac{d-1}{2} - \ell)^2$ where σ_n means $\sigma_n^\pm$ if $d = 2n+1$.*

To deal with the heat operator $e^{-t\Delta_k}$, we follow the discussion in the section 2 of [2] or the section 3 of [19]. Let us denote by $\Omega_K = -\sum E_i^2$ the normalized Casimir operator of K. Recalling (2.5), let Δ_G denote the corresponding left invariant Laplace operator over G; that is,

$$\Delta_G = -\Omega + 2\Omega_K = -\sum E_i^2 - \sum E_j^2. \tag{2.7}$$

Using the following well known formula (for instance, see (A.10) of [1])

$$\tau_k(\Omega_K) = \langle \mu_k + \rho_K, \mu_k + \rho_K \rangle - \langle \rho_K, \rho_K \rangle$$

where ρ_K denotes the half sum of the positive roots of $K = \mathrm{SO}(d)$, one can show that

$$\tau_k(\Omega_K) = \lambda_k \mathrm{Id}_{V_{\tau_k}} \qquad \text{with} \quad \lambda_k = (d-k)k.$$

(This formula is even true for τ_n with $d = 2n$.) By (2.6) and (2.7),

$$\Delta_k = Q_{\tau_k} (\Delta_G \otimes \mathrm{Id}_{V_{\tau_k}}) Q_{\tau_k} - 2\lambda_k \mathrm{Id}_{V_{\tau_k}}.$$

Now let $e^{-t\Delta_G}$ denote the heat semi-group operator given by a smooth kernel P_t,

$$e^{-t\Delta_G} f(g_1) = \int_G P_t(g_2^{-1} g_1) f(g_2) dg_2, \qquad \text{for} \quad f \in L^2(G),\ g_1 \in G.$$

Hence, the heat semi-group operator of Δ_k satisfies

$$e^{-t\Delta_k} = e^{2\lambda_k t} \cdot Q_{\tau_k}(e^{-t\Delta_G} \otimes \mathrm{Id}_{V_{\tau_k}})Q_{\tau_k}, \tag{2.8}$$

which implies

$$e^{-t\Delta_k}(g_2^{-1}g_1) = e^{2\lambda_k t} \cdot \int_{K\times K} \tau_k(k_2)P_t(k_1 g_2^{-1} g_1 k_2)\tau_k(k_1)\, dk_1 dk_2.$$

Therefore, the kernel $e^{-t\Delta_k}(g_1, g_2) := e^{-t\Delta_k}(g_2^{-1}g_1)$ satisfies the following covariance relation

$$e^{-t\Delta_k}(g_1k_1, g_2k_2) = \tau_k(k_1)^{-1}e^{-t\Delta_k}(g_1, g_2)\tau_k(k_2) \qquad \text{for} \quad k_1, k_2 \in K. \tag{2.9}$$

For a fixed $t > 0$, by Lemma 2.3 in [2], $e^{-t\Delta_G}$ belongs to the Harish-Chandra L^p-Schwartz space $\mathcal{C}^p(G)$ for any $p > 0$. Here $\mathcal{C}^p(G)$ is the space of all functions $f \in C^\infty(G)$ such that

$$\sup_{g\in G} (1 + \sigma(g))^m \Psi(g)^{-\frac{2}{p}} |D_1 D_2 f(g)| < \infty \qquad \text{for any} \quad m \geq 0, \quad D_1, D_2$$

where $\sigma(g)$ is the geodesic distance between the cosets eK and gK in G/K,

$$\Psi(g) = \int_K e^{-\rho(H(gk))}\, dk$$

for the Iwasawa decomposition $gk = K(gk)\exp(H(gk))N(gk)$, and D_1, D_2 denote the right, left invariant differential operators, respectively. Let us remark that $\mathcal{C}^p(G) \subseteq \mathcal{C}^{p'}(G)$ if $p \leq p'$ (see p.4 in [8]). Now we can conclude

Proposition 2.2. *For any $t > 0$, the heat kernel $e^{-t\Delta_k}$ belongs to $(\mathcal{C}^p(G) \otimes \mathrm{End}(V_{\tau_k}))^{K\times K}$ for any $p > 0$ and $e^{-t\Delta_k}$ satisfies the covariance relation in (2.9).*

Now let $h_t^k := \mathrm{tr}(e^{-t\Delta_k})$ where tr is given over V_{τ_k}. Then h_t^k belongs to $\mathcal{C}^p(G)$ for $p > 0$ by Proposition 2.2. Hence one can define

$$\Theta_{\sigma,\lambda}(h_t^k) = \mathrm{Tr}\, \pi_{\sigma,\lambda}(h_t^k) = \mathrm{Tr} \int_G h_t^k(g)\pi_{\sigma,\lambda}(g)\, dg.$$

For a given unitary representation π of G, we define its matrix block of π corresponding to $P_\tau \in \mathrm{Hom}_K(\pi|_K, \tau)$ by

$$\Phi_\tau^\pi(g) := P_\tau \pi(g^{-1})P_\tau^* \qquad \text{for} \quad g \in G.$$

We note that $\Phi_\tau^\pi(g) \in \mathrm{End}(V_\tau)$ for a fixed $g \in G$ and

$$\Phi_\tau^\pi(k_1 g k_2) = \tau(k_2)^{-1}\Phi_\tau^\pi(g)\tau(k_1)^{-1} \qquad \text{for} \quad k_1, k_2 \in K.$$

This is an τ-spherical function on G on which the normalized Casimir operator Ω in (2.5) acts by its infinitesimal character. We refer to the section VIII 4-6 in [14] for more details of these facts. Hence, by Proposition 2.1, for $\tau = \tau_k, \pi = \pi_{\sigma_\ell,\lambda}$ we have

$$\int_G e^{-t\Delta_k}(g_2^{-1}g_1)\Phi_\tau^\pi(g_2)\, dg_2 = \exp\Big(- t\big(\lambda^2 + \big(\frac{(d-1)}{2} - \ell\big)^2\big)\Big)\, \Phi_\tau^\pi(g_1) \tag{2.10}$$

if $[\tau_k|_M : \sigma_\ell] \neq 0$ and both sides of (2.10) vanish if $[\tau_k|_M : \sigma_\ell] = 0$. Taking the trace over V_τ and putting $g_1 = e$, we get

$$\dim V_\tau \int_G h_t^k(g_2^{-1})\, \widetilde{\Phi}_\tau^\pi(g_2)\, dg_2 = \exp\Big(- t\big(\lambda^2 + \big(\frac{(d-1)}{2} - \ell\big)^2\big)\Big)\, \widetilde{\Phi}_\tau^\pi(e) \tag{2.11}$$

by the orthogonality relations for the matrix elements of τ where $\widetilde{\Phi}^\pi_\tau := \operatorname{tr}\Phi^\pi_\tau$. On the other hand,

$$\operatorname{Tr}\pi(h^k_t) = \int_G h^k_t(g^{-1})\widetilde{\Phi}^\pi_\tau(g)\,dg \qquad \text{for} \quad \pi = \pi_{\sigma,\lambda}. \tag{2.12}$$

Comparing (2.11) and (2.12) and noting $\widetilde{\Phi}^\pi_\tau(e) = \dim V_\tau$, we obtain

Proposition 2.3.

$$\Theta_{\sigma_\ell,\lambda}(h^k_t) = \begin{cases} \exp\Big(-t(\lambda^2 + (\frac{(d-1)}{2} - \ell)^2)\Big) & \text{if} \quad [\tau_k|_M : \sigma_\ell] \neq 0 \\ 0 & \text{if} \quad [\tau_k|_M : \sigma_\ell] = 0 \end{cases}.$$

3. Spectral decomposition for hyperbolic manifolds with cusps

3.1. **Hodge Laplacian for hyperbolic manifolds with cusps.** Let us choose a unitary representation χ of Γ on a finite-dimensional hermitian vector space V_χ. We now consider the right quasi-regular representation R_χ on

$$\mathrm{H}_\chi := \big\{\, \phi : G \to V_\chi \,|\, \phi(\gamma x) = \chi(\gamma)\phi(x) \ \text{ for } \gamma \in \Gamma, x \in G, \ \ |\phi\,| \in L^2(\Gamma\backslash G) \,\big\}$$

given by $(R_\chi(x)\phi)(y) = \phi(yx)$. As in [29], this representation R_χ of G on H_χ decomposes into a discrete part and a continuous part. That is,

$$R_\chi = R^d_\chi \oplus R^c_\chi \qquad \text{acts on} \qquad \mathrm{H}_\chi = \mathrm{H}^d_\chi \oplus \mathrm{H}^c_\chi.$$

The action R^d_χ on H^d_χ is a direct sum of irreducible representations, each of them occurring with finite multiplicity and the action of R^c_χ on H^c_χ is a direct integral, with no irreducible sub-representations, of principle series.

For a test function $h \in \mathcal{C}^p(G)$ with $0 < p < 1$, which is of right K-finite, the induced representation $R^d_\chi(h)$ is of trace class and

$$\operatorname{Tr} R^d_\chi(h) = \operatorname{Tr}\int_G h(g)\,R^d_\chi(g)\,dg = \sum_{\pi\in\widehat{G}} m_\chi(\pi)\operatorname{Tr}\pi(h) \tag{3.1}$$

where $m_\chi(\pi)$ denotes the multiplicity of $\pi \in \widehat{G}$ in H^d_χ.

Let us recall that a d-dimensional noncompact hyperbolic manifold with cusps is given by

$$X_\Gamma = \Gamma\backslash G/K = \Gamma\backslash \mathrm{SO}_0(d,1)/\mathrm{SO}(d)$$

where Γ is a cofinite discrete subgroup of $G = \mathrm{SO}_0(d,1)$ satisfying the conditions imposed in the introduction. The vector bundle E^k_χ over X_Γ of k-forms twisted by χ is given by

$$E^k_\chi = V_\chi \times_\chi G \times_{\tau_k} V_{\tau_k}.$$

The Hodge Laplacian Δ_k acting on the space of sections of V_{τ_k} over G/K can be naturally pushed down to a differential operator acting on $C^\infty_0(X_\Gamma, E^k_\chi)$. By abuse of notation, we use the same notation Δ_k to denote its self-adjoint extension on

$$L^2(X_\Gamma, E^k_\chi) = \{\ |f| \in L^2(X_\Gamma, V_{\tau_k}) \ | \ f(\gamma x) = \chi(\gamma) f(x) \ \text{ for } \gamma \in \Gamma\ \},$$

which consists of the τ_k-isotypic component of H_χ. In general, the operator Δ_k on $L^2(X_\Gamma, E^k_\chi)$ has discrete spectrum $\sigma_p(\Delta_k)$ as well as continuous spectrum $\sigma_c(\Delta_k)$. The continuous spectrum of Δ_k is mainly controlled by the scattering operators $C^k_\chi(\sigma_k, s)$ and $C^k_\chi(\sigma_{k-1}, s)$, which will be explained in the next subsection, for purely imaginary numbers $s = i\lambda \in \mathbb{C}$. These scattering operators have the matrix forms of size $d_c(\chi)$ where

$$d_c(\chi) = \sum_{j=1}^{\kappa} d_j(\chi). \tag{3.2}$$

Here $d_j(\chi)$ denotes the dimension of the maximal subspace of V_χ over which $\chi|_{\Gamma\cap P_j}$ acts trivially for $P_j \in \mathfrak{P}_\Gamma$. When $d = 2n+1$, the scattering operator $C^n_\chi(\sigma_n, s)$ has the size $2\,d_c(\chi)$ since $\sigma_n^\pm$ is un-ramified.

3.2. **Scattering operators and Maass-Selberg relation.** Let

$$L^2(M) = \sum_{\sigma\in\widehat{M}} \oplus\, d(\sigma) H_\sigma, \qquad R_M = \sum_{\sigma\in\widehat{M}} \oplus\, d(\sigma)\sigma$$

be the decomposition of the right regular representation R_M of M on $L^2(M)$ where $d(\sigma) = \dim H_\sigma$. A similar induction procedure to the principal series representation starting with R_M instead of $\sigma \in \widehat{M}$ gives rise to a unitary representation of G,

$$\sum_{\sigma\in\widehat{M}} \oplus\, \pi(\sigma,\lambda) \qquad \text{acts on} \qquad \sum_{\sigma\in\widehat{M}} \oplus\, \mathcal{H}(\pi(\sigma,\lambda))$$

where

$$\pi(\sigma,\lambda) = \begin{cases} d(\sigma)\pi_{\sigma,\lambda} & \text{if} \quad w\sigma = \sigma, \\ d(\sigma)\pi_{\sigma,\lambda} \oplus d(w\sigma)\pi_{w\sigma,\lambda} & \text{if} \quad w\sigma \neq \sigma. \end{cases}$$

Here w is the nontrivial element in $W(G,A)$. Now for $P_j \in \mathfrak{P}_\Gamma$ with the corresponding decomposition $P_j = N_jA_jM_j$ where $P_j = x_jP_0x_j^{-1}$, $N_j = x_jNx_j^{-1}$, $A_j = x_jAx_j^{-1}$, $M_j = x_jMx_j^{-1}$ for certain $x_j \in K$, the above definitions carry over to each M_j with obvious changes of notation such as $\pi(\sigma_j,\lambda_j)$ for $1 \le j \le \kappa$. For $\tau \in \widehat{K}$ with $[\tau|_M : \sigma_j] \neq 0$, let us observe

$$\mathcal{H}(\sigma_j,\tau) := (\mathcal{H}_{\sigma_j,\lambda_j} \otimes V_\tau)^K \quad\cong\quad (H_{\sigma_j}\otimes V_\tau)^M,$$

and that the τ-isotypic component of $\mathcal{H}(\pi(\sigma_j,\lambda_j))$ can be identified with the $d(\sigma_j)$ copies of $(H_{\sigma_j}\otimes V_\tau)^M$ if $w\sigma_j = \sigma_j$, or $d(\sigma_j)$ copies of $((H_{\sigma_j} \oplus H_{w\sigma_j})\otimes V_\tau)^M$ if $w\sigma_j \neq \sigma_j$. The second case happens if and only if $\tau = \tau_n, \sigma = \sigma_n^\pm$ with $d = 2n+1$.

For $P = P_j \in \mathfrak{P}_\Gamma$ and $\Phi \in V_P \otimes \mathcal{H}(\sigma,\tau)$ where V_P denotes the maximal invariant subspace of V_χ under $\chi|_{\Gamma\cap P}$, the Eisenstein series attached to Φ is defined as

$$E(P,\Phi,s,x) := \sum_{\gamma\in\Gamma/\Gamma\cap P} \chi(\gamma) e^{(s+\rho)(H(\gamma^{-1}x))}\Phi(\gamma^{-1}x) \qquad \text{for} \quad \mathrm{Re}(s) > \frac{d-1}{2}$$

where $H(x) = H_j(x)$ is given by the decomposition $x = N_j(x)\exp(H_j(x))K(x)$. This is absolutely and uniformly convergent on compact sets in the half plane $\mathrm{Re}(s) > \frac{d-1}{2}$, and extends meromorphically to $\mathbb{C}$. These facts can be proved as in [11], [29]. For $P_i, P_j \in \mathfrak{P}_\Gamma$, the constant term of $E(P_i,\Phi,s,x)$ along P_j is defined by

$$E_{P_j}(P_i,\Phi,s,x) = \frac{1}{\mathrm{vol}(\Gamma\cap N_j\backslash N_j)} \int_{\Gamma\cap N_j\backslash N_j} E(P_i,\Phi,s,nx)\,dn$$

and has the following expression along P_j,

$$E_{P_j}(P_i,\Phi,s,x)=\sum_{w\in W(A_i,A_j)} e^{(ws+\rho)(H_j(x))}\big(C^\tau_{ji}(w,s)\Phi\big)(x)$$

where $W(A_i,A_j)$ denotes the set of all bijections $w:A_i\to A_j$ defined by $wa_i=xa_ix^{-1}$ for $x\in K$ and

$$C^\tau_{ji}(w,s):V_{P_i}\otimes\mathcal{H}(\sigma_i,\tau)\longrightarrow V_{P_j}\otimes\mathcal{H}(\sigma_j,\tau),\qquad w\in W(A_i,A_j).$$

Now combining the operators $C^\tau_{ji}(x_iwx_j^{-1},x_j\cdot s)$ with the nontrivial element $w\in W(A,A)$ defines the scattering operator

$$C^\tau_\chi(\sigma,s)\qquad\text{on}\qquad \mathcal{H}_\chi(\sigma,\tau):=\sum_{j=1}^{\kappa}\oplus\ V_{P_j}\otimes\mathcal{H}(\sigma_j,\tau).$$

When $\tau=\tau_k$, we denote $C^{\tau_k}_\chi(\sigma,s)$ by $C^k_\chi(\sigma,s)$ for simplicity. In a natural way, we see that

$$\mathcal{H}_\chi(\sigma,\tau):=\sum_{j=1}^{\kappa}\oplus\ V_{P_j}\otimes\mathcal{H}(\sigma_j,\tau)\ \cong\ V_c\otimes\mathcal{H}(\sigma,\tau)\qquad\text{where}\qquad V_c:=\sum_{j=1}^{\kappa}\oplus V_{P_j}.$$

The scattering operator has a meromorphic extension over $\mathbb{C}$ and it satisfies the well-known functional equations

$$C^\tau_\chi(\sigma,s)C^\tau_\chi(\sigma,-s)=\mathrm{Id},\qquad C^\tau_\chi(\sigma,s)^*=C^\tau_\chi(\sigma,\bar{s}). \tag{3.3}$$

Now we analyze $R^c_\chi(h)$ for $h\in\mathcal{C}^p(G)$ $(0<p<1)$ assuming that h is of fixed τ-type. We also assume that $\Theta_{\sigma,\lambda}(h)=\Theta_{w\sigma,w\lambda}(h)$ if $w\sigma\neq\sigma$. (The function h^k_t defined in the previous section satisfies these conditions.) Let us choose an orthonormal basis $\{\Phi_{mn}=v_m\otimes\xi_n\}$ of $\mathcal{H}_\chi(\sigma,\tau)$. We put

$$E(s,x):=\sum_{m,n}E(\Phi_{mn},s,x),$$

where $E(\Phi_{mn},s,x)$ is defined as the usual Eisenstein series $E(P,\Phi,s,x)$. Then the kernel $K^c(h:x,y)$ of $R^c_\chi(h)$ on H^c_χ is given by

$$K^c(h:x,y)=\sum_{\sigma\in\widehat{M}}[\tau|_M:\sigma]\,\frac{d(\sigma)}{4\pi}\int_{-\infty}^{\infty}\pi_\chi(\sigma,\lambda)(h)\,E(i\lambda,x)\otimes E(i\lambda,y)^*\,d\lambda$$

where $\pi_\chi(\sigma,\lambda)$ is the representation of G on $\mathcal{H}_\chi(\sigma,\tau)$ defined by the $\pi(\sigma_j,\lambda_j)$'s.

For $P_j\in\mathfrak{P}_\Gamma$, the subset $\mathcal{C}_j(u)=N_jA_j(u)K\subset G$ is called a cylindrical domain where

$$A_j(u)=\{\,a_t\in A_j\,|\,a_t=\exp(tH_j),\quad t\geq u\,\}.$$

Then there is a $u_0\gg 0$ such that the sets $C_j(u):=p(\mathcal{C}_j(u))\subset X_\Gamma=\Gamma\backslash G/K$ are disjoint to each other for $u\geq u_0$ and $1\leq j\leq\kappa$ where $p:G\to X_\Gamma$ denotes the natural projection. The measure in (2.3) induces the metric

$$dt^2+e^{-2t}dn^2$$

over $C_j(u)$ where dn^2 is the flat metric over $(\Gamma\cap N_j)\backslash N_j$. We put $X_\Gamma(u):=X_\Gamma-\cup_{j=1}^{\kappa}C_j(u)$. Now we have an expansion formula of $\int_{X_\Gamma(u)}\mathrm{tr}\big(K^c(h:x,x)\big)\,dx$ as $u\to\infty$ in the following theorem which can be proved as in the section 6 of [29].

Theorem 3.1. (Maass-Selberg relation) *For $u \geq u_0 \gg 0$, we have*

$$\begin{aligned}&\int_{X_\Gamma(u)} \operatorname{tr}\big(K^c(h:x,x)\big)\,dx\\ &= \sum_{\sigma\in\widehat{M}} [\tau|_M:\sigma]\,\frac{d(\sigma)}{4\pi}\Big(2u\int_{-\infty}^{\infty}\Theta_{\sigma,\lambda}(h)\,d\lambda - \int_{-\infty}^{\infty}\Theta_{\sigma,\lambda}(h)\operatorname{tr}\big(C^\tau_\chi(\sigma,-i\lambda)\partial_{i\lambda}C^\tau_\chi(\sigma,i\lambda)\big)\,d\lambda\\ &\qquad + \pi\,\Theta_{\sigma,0}(h)\operatorname{tr}\big(C^\tau_\chi(\sigma,0)\big)\Big) + \mathrm{O}(u^{-1}).\end{aligned}$$

4. Selberg trace formula

4.1. **Trace formula.** For $0<p<1$ the Selberg trace applied to $h \in \mathcal{C}^p(G)$ has the following form,

$$\operatorname{Tr} R^d_\chi(h) = I_\chi(h) + H_\chi(h) + U_\chi(h) + W_\chi(h) + S_\chi(h) + J_\chi(h) \tag{4.1}$$

where the left-hand side has the form in (3.1). The terms on the right-hand side are explained as follows. Here I_χ, H_χ, U_χ are given by the identity, hyperbolic, unipotent orbital integrals respectively. These are invariant tempered distributions on G, which were fully analyzed in [25].

First, for $I_\chi(h)$ we have

$$I_\chi(h) = \dim V_\chi \cdot \operatorname{vol}(\Gamma\backslash G)\cdot h(e).$$

By the Plancherel theorem,

$$h(e) = \sum_{\omega\in\widehat{G}_d} d(\omega)\Theta_\omega(h) + \sum_{\sigma\in\widehat{M}} \frac{1}{4\pi}\int_{-\infty}^{\infty}\Theta_{\sigma,\lambda}(h)\,p(\sigma,\lambda)\,d\lambda$$

where $d(\omega)$ denotes the formal degree of $\omega\in\widehat{G}_d$ and $p(\sigma,\lambda)$ denotes the Plancherel measure. Let us recall that for $G=\mathrm{SO}_0(2n+1,1)$ there is no discrete series so that there are no terms from $\widehat{G}_d$ in the above formula. For $G=\mathrm{SO}_0(2n,1)$, the discrete series may give a nontrivial contribution in general. For $h=h^k_t$, we can see that this contribution is nontrivial only when $k=n$, $d=2n$ and it is the harmonic part of Δ_n in $L^2(G,V_{\tau_n})$ by Theorem 3.2 in [23]. Repeating the argument for the τ-spherical function defined for $\omega\in\widehat{G}_d$ as in the subsection 2.3, we see that $\Theta_\omega(h^k_t)$ is a nonzero constant only when $k=n$, $d=2n$ and $\Theta_\omega(h^k_t)=0$ otherwise. Now we have

$$\begin{aligned}I_\chi(h^k_t) = \dim V_\chi\cdot\operatorname{vol}(\Gamma\backslash G)\cdot\Big(&\tilde{\delta}_n(k)c(\tau_n)\\ &+ \sum_{[\tau_k|_M:\sigma_\ell]\neq 0}\frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-t(\lambda^2+(\frac{(d-1)}{2}-\ell)^2)}\,p(\sigma_\ell,\lambda)\,d\lambda\Big)\end{aligned} \tag{4.2}$$

where $\tilde{\delta}_n(k)=1$ if $k=n$, $d=2n$, $\tilde{\delta}_n(k)=0$ otherwise, and $c(\tau_n)$ is a constant only depending on τ_n.

By Theorem 3.1 in [17] and taking care of the normalization, the Plancherel measure corresponding to $(\pi_{\sigma_k,\lambda}, \mathcal{H}_{\sigma_k,\lambda})$ is given by

$$
\begin{aligned}
(4.3)\qquad p(\sigma_k,\lambda) &= \pi\, 2^{-4(n-\frac{1}{2})}\Gamma(n+\frac{1}{2})^{-2}\, d(\sigma_k) \\
&\quad \prod_{j=1}^{k}\big(\lambda^2+(n-j+1)^2\big)\prod_{j=k+1}^{n}\big(\lambda^2+(n-j)^2\big) \qquad \text{if}\quad d=2n+1,\\
p(\sigma_k,\lambda) &= \pi\, 2^{-4(n-1)}\Gamma(n)^{-2}\, d(\sigma_k)\, \tanh(\pi\lambda)\\
&\quad \lambda\prod_{j=1}^{k}\big(\lambda^2+(n-j+\frac{1}{2})^2\big)\prod_{j=k+1}^{n-1}\big(\lambda^2+(n-j-\frac{1}{2})^2\big) \qquad \text{if}\quad d=2n
\end{aligned}
$$

where σ_n means $\sigma_n^{\pm}$ when $d=2n+1$.

The term $H_\chi(h)$ is given by

$$
(4.4)\qquad H_\chi(h) = \sum_{\gamma\in\Gamma_{\mathrm{hyp}}} \operatorname{tr}\chi(\gamma)\ \cdot \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\ \cdot \int_{G_\gamma\backslash G} h(g^{-1}\gamma g)\, d(G_\gamma g)
$$

where Γ_{hyp} denotes the set of the Γ-conjugacy classes of the hyperbolic elements in Γ, and Γ_γ, G_γ denote the centralizers of γ in Γ, G respectively. We may assume that a hyperbolic element $\gamma\in\Gamma$ has the form $a_\gamma m_\gamma \in A^+M$ where $A^+=\{e^{tH}, t>0\}$. By the section 6 in [28], we have

$$
\begin{aligned}
&\operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\ \cdot \int_{G_\gamma\backslash G} h(g^{-1}\gamma g)\, d(G_\gamma g)\\
(4.5)\qquad &= \sum_{\sigma\in\widehat{M}} l(C_\gamma) j(\gamma)^{-1} D(a_\gamma m_\gamma)^{-1}\, \overline{\operatorname{tr}\sigma(m_\gamma)}\, \frac{1}{2\pi}\int_{-\infty}^{\infty} \Theta_{\sigma,\lambda}(h) e^{-il(C_\gamma)\lambda}\, d\lambda
\end{aligned}
$$

where $l(C_\gamma)$ denotes the length of the closed geodesic determined by γ, $j(\gamma)$ denotes the positive integer such that $\gamma=\gamma_0^{j(\gamma)}$ with a primitive γ_0, and

$$
D(a_\gamma m_\gamma) = a_\gamma^{\rho}\, \big|\det\big(\operatorname{Ad}(a_\gamma m_\gamma)^{-1} - \operatorname{Id}|_{\mathfrak{n}}\big)\,\big|.
$$

For $h=h_t^k$, we have

$$
\begin{aligned}
(4.6)\qquad H_\chi(h_t^k) =& \frac{1}{\sqrt{4\pi t}} \sum_{\gamma\in\Gamma_h} \operatorname{tr}\chi(\gamma) \sum_{[\tau_k|_M:\sigma_\ell]\neq 0} l(C_{\gamma_0})\\
&\big|\det\big(\operatorname{Ad}(a_\gamma m_\gamma)^{-1} - \operatorname{Id}|_{\mathfrak{n}}\big)\,\big|^{-1}\, \overline{\operatorname{tr}\sigma_\ell(m_\gamma)}\, e^{-\frac{l(C_\gamma)^2}{4t}}\, e^{-t\left(\frac{(d-1)}{2}-\ell\right)^2}\, e^{-\frac{(d-1)l(C_\gamma)}{2}}
\end{aligned}
$$

by Proposition 2.3, the Fourier integral of the Gaussian and $a_\gamma^{\rho} = e^{\frac{(d-1)l(C_\gamma)}{2}}$.

The terms $U_\chi(h), W_\chi(h)$ will be discussed later. The scattering term $S_\chi(h)$ and the residual term $J_\chi(h)$ have the following form,

$$
\begin{aligned}
S_\chi(h) =& \frac{1}{4\pi}\sum_{\tau\in\widehat{K}}\sum_{\sigma\in\widehat{M}} [\tau|_M:\sigma]\, d(\sigma) \int_{-\infty}^{\infty} \operatorname{Tr}\big(\pi_\chi(\sigma,\lambda)(h) C_\chi^\tau(\sigma,-i\lambda)\partial_{i\lambda} C_\chi^\tau(\sigma,i\lambda)\big)\, d\lambda,\\
J_\chi(h) =& -\frac{1}{4}\sum_{\tau\in\widehat{K}}\sum_{\sigma\in\widehat{M}} [\tau|_M:\sigma]\, d(\sigma)\, \operatorname{Tr}\big(\pi_\chi(\sigma,0)(h) C_\chi^\tau(\sigma,0)\big).
\end{aligned}
$$

For $h = h_t^k$, these term are

$$S_\chi(h) = \frac{1}{4\pi} \sum_{[\tau_k|_M:\sigma_\ell]\neq 0} d(\sigma_\ell) \int_{-\infty}^{\infty} e^{-t(\lambda^2+(\frac{(d-1)}{2}-\ell)^2)} \mathrm{tr}\big(C_\chi^k(\sigma_\ell,-i\lambda)\partial_{i\lambda}C_\chi^k(\sigma_\ell,i\lambda)\big)\,d\lambda, \tag{4.7}$$

$$J_\chi(h) = -\frac{1}{4} \sum_{[\tau_k|_M:\sigma_\ell]\neq 0} d(\sigma_\ell)\, e^{-t(\frac{(d-1)}{2}-\ell)^2} \mathrm{tr}\big(C_\chi^k(\sigma,0)\big), \tag{4.8}$$

which are the finite terms as $|u| \to \infty$ on the right-hand side of the Maass-Selberg relation in Theorem 3.1 when $h = h_t^k$.

4.2. **Unipotent terms.** By the computation in [20], the terms $U_\chi(h)$ and $W_\chi(h)$ are given by the sum over $P = NAM \in \mathfrak{P}_\Gamma$ of the following term

$$\mathrm{vol}(\Gamma_P\backslash N(P)) \lim_{s\to 0} \frac{d}{ds}\Big(s\,\zeta_P(s,\chi)T_P(h,s)\Big) \tag{4.9}$$

under our normalization. Here the Epstein type zeta function $\zeta_P(s,\chi)$ is defined by

$$\zeta_P(s,\chi) = \sum_{\eta\in\Gamma_P,\eta\neq e} \mathrm{tr}\,\chi(\eta)|X_\eta|^{-(d-1)(s+1)} \qquad \text{for} \quad \mathrm{Re}(s) > 0$$

where $\eta = \exp(X_\eta)$ and $|X_\eta|^2 = \langle X_\eta, X_\eta\rangle$. The other term $T_P(h,s)$ is given by

$$T_P(h,s) = \frac{1}{A(\mathfrak{n})} \int_N \int_K h(knk^{-1})|\log n|^{(d-1)s}\,dk\,dn$$

where $A(\mathfrak{n})$ is the volume of the unit sphere in $\mathfrak{n}$. By section 1 of [20] (and section 7 of [29]), we know that $s \mapsto T_P(h,s)$ is holomorphic over a certain strip containing the imaginary axis. Let us remark that the condition (1.1) is used in the derivation of (4.9) and will be used in the forthcoming analysis of $\zeta_P(s,\chi)$. Now let us observe that $\chi|_{\Gamma_P}$ decomposes into one-dimensional representations χ_θ's of Γ_P (since Γ_P is abelian by (1.1)) such that

$$\chi_\theta(\eta) = \exp\big(2\pi i(n_1\theta_1 + \ldots + n_{d-1}\theta_{d-1})\big) \qquad \text{for} \quad \eta = \prod_{j=1}^{d-1} \eta_j^{n_j}$$

where $\{\eta_i\}$ denotes a fixed basis of Γ_P. For $P \in \mathfrak{P}_\Gamma$, we decompose

$$V = V_P \oplus {V_P}^\perp$$

where $V_P \subset V$ is the maximal subspace over which $\chi|_{\Gamma_P}$ acts trivially, so that χ decomposes into a direct sum of Id_{V_P} and χ_θ's with nontrivial $\theta = (\theta_1, \ldots, \theta_{d-1})$, that is, one of θ_i is not an integer.

Proposition 4.1. *The Epstein type zeta function*

$$\zeta_P(s,\chi_\theta) := \sum_{\eta\in\Gamma_P,\eta\neq e} \chi_\theta(\eta)|X_\eta|^{-(d-1)(s+1)}$$

has a meromorphic extension over $\mathbb{C}$. This meromorphic function is entire if θ is nontrivial and has a simple pole at $s = 0$ if θ is trivial.

Proof. Since $\zeta_P(s,\chi_\theta)$ is absolutely convergent for $\mathrm{Re}(s) > 0$, it is enough to consider a meromorphic extension of

$$\zeta_P(s,\chi_\theta) = \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\Big(\sum_{\eta\in\Gamma_P,\eta\neq e}\chi_\theta(\eta)e^{-t|X_\eta|^2}\Big)\,dt \qquad \text{with} \quad z=\frac{(d-1)}{2}(s+1),$$

over the left half plane $\mathrm{Re}(s)\leq 0$. By a standard argument, one can obtain such a meromorphic extension over $\mathbb{C}$ if we have the asymptotic expansion of

$$\sum_{\eta\in\Gamma_P,\eta\neq e}\chi_\theta(\eta)e^{-t|X_\eta|^2} \qquad \text{as} \quad t\to 0. \tag{4.10}$$

To this end, we recall the Jacobi type identity

$$\sum_{\eta\in\Gamma_P}\chi_\theta(\eta)e^{-t|X_\eta|^2} = \mathrm{vol}(\Gamma_P\backslash N(P))^{-1}\big(\frac{\pi}{t}\big)^{\frac{(d-1)}{2}}\sum_{\lambda_k}e^{-\frac{\lambda_k}{4t}} \tag{4.11}$$

where λ_k denotes the eigenvalues of the Laplacian Δ_θ acting on the space of the sections of the flat vector bundle defined by χ_θ over $\Gamma_P\backslash N(P)$. The equality (4.11) follows by putting $t=(4s)^{-1}$ at the following equality

$$\mathrm{Tr}(e^{-s\Delta_\theta}) = \int_{\Gamma_P\backslash N(P)}(4\pi s)^{-\frac{d-1}{2}}\Big(\sum_{\eta\in\Gamma_P}\chi_\theta(\eta)e^{-\frac{d(n,\eta n)^2}{4s}}\Big)\,dn$$

where $d(n,\eta n)$ denotes the Euclidean distance given by the normalized Cartan-Killing form, which equals to $|X_\eta|$. Then Δ_θ has the zero eigenvalue if and only if θ is trivial. Hence

$$\sum_{\eta\in\Gamma_P,\eta\neq e}\chi_\theta(\eta)e^{-t|X_\eta|^2} = -1+R(t)$$

where $R(t)=O(e^{-\frac{c}{t}})$ for a certain $c>0$ as $t\to 0$ if θ is nontrivial and $t^{-\frac{(d-1)}{2}}+O(e^{-\frac{c}{t}})$ as $t\to 0$ if θ is trivial. It follows that the meromorphic extension of $\zeta_P(s,\chi_\theta)$ is entire if θ is nontrivial and has a simple pole at $s=0$ if θ is trivial.

□

Now we have

$$\zeta_P(s,\chi) = d_P(\chi)\cdot\sum_{\eta\in\Gamma_P,\eta\neq e}|X_\eta|^{-(d-1)(s+1)} + \sum_\theta\sum_{\eta\in\Gamma_P,\eta\neq e}\chi_\theta(\eta)|X_\eta|^{-(d-1)(s+1)} \tag{4.12}$$

where $d_P(\chi)=\dim V_P$ and the second sum runs over the nontrivial θ. The first and second sums on the right-hand side of (4.12) have a simple pole, and is regular at $s=0$, respectively, by Proposition 4.1. Therefore we conclude

$$\lim_{s\to 0}\frac{d}{ds}\Big(s\,\zeta_P(s,\chi)T_P(h,s)\Big) = d_P(\chi)\,\big(C_PT_P(h)+R_PT_P'(h)\big)+\widetilde{C}_PT_P(h) \tag{4.13}$$

where C_P, R_P denote, respectively, the constant term and the residue of the ordinary Epstein zeta function at $s=0$, $\widetilde{C}_P$ denotes the sum of the constant terms of $\zeta_P(s,\chi_\theta)$ with nontrivial θ at $s=0$, and

$$T_P(h) = \frac{1}{A(\mathfrak{n})}\int_N\int_K h(knk^{-1})\,dk\,dn,$$
$$T_P'(h) = \frac{(d-1)}{A(\mathfrak{n})}\int_N\int_K h(knk^{-1})\log|\log n|\,dk\,dn.$$

The term $U_\chi(h)$ is the sum over $P \in \mathfrak{P}_\Gamma$ of the invariant part of the right side of (4.13), that is,

$$U_\chi(h) = \sum_{P_j \in \mathfrak{P}_\Gamma} \mathrm{vol}(\Gamma_{P_j} \backslash N(P_j)) \left(d_{P_j}(\chi) C_{P_j} + \widetilde{C}_{P_j} \right) T_{P_j}(h) \tag{4.14}$$

with

$$T_{P_j}(h) = \frac{1}{A(\mathfrak{n})} \sum_{\sigma \in \widehat{M}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \Theta_{\sigma,\lambda}(h)\, d\lambda \tag{4.15}$$

by the section 6 in [28]. The remaining part is

$$W_\chi(h) = \sum_{P_j \in \mathfrak{P}_\Gamma} \mathrm{vol}(\Gamma_{P_j} \backslash N(P_j))\, d_{P_j}(\chi) R_{P_j} T'_{P_j}(h).$$

By the computation in [3],

$$\mathrm{vol}(\Gamma_P \backslash N(P))\, R_P \frac{(d-1)}{A(\mathfrak{n})} = 1 \qquad \text{for } \ P \in \mathfrak{P}_\Gamma$$

under our normalization. Hence

$$W_\chi(h) = d_c(\chi) \int_N \int_K h(knk^{-1}) \log|\log n|\, dk\, dn. \tag{4.16}$$

5. Computation of the weighted orbital integral

5.1. **Weighted orbital integral.** The weighted orbital integral given in (4.16)

$$W_\chi(h) = d_c(\chi) \int_N \int_K h(knk^{-1}) \log|\log n|\, dk\, dn \tag{5.1}$$

is a non-invariant tempered distribution. To explain this, let us recall that the intertwining operator

$$J_{\bar{P}|P}(\sigma,\lambda)\phi := \int_{\bar{N}} \phi(x\bar{n}) d\bar{n} \ : \ \mathcal{H}_{\sigma,\lambda}(P) \to \mathcal{H}_{\sigma,\lambda}(\bar{P})$$

where the notation $\mathcal{H}_{\sigma,\lambda}(P)$ denotes the principal series representation with its dependence on P. The restriction to K defines an isomorphism from $\mathcal{H}_{\sigma,\lambda}(P)$ to

$$L^2(K, H_\sigma) := \{\, f : K \to H_\sigma \mid f(mk) = \sigma(m) f(k),\ |f| \in L^2(K) \,\}.$$

By this isomorphism, $J_{\bar{P}|P}(\sigma,\lambda)$ can be regarded as a family of operators acting on $L^2(K, H_\sigma)$. Let

$$J_P(\sigma, \lambda : h) = -\mathrm{Tr}(\pi_{\sigma,\lambda}(h)\, J_{\bar{P}|P}(\sigma,\lambda)^{-1} \partial_{i\lambda} J_{\bar{P}|P}(\sigma,\lambda))$$

where $\partial_{i\lambda}$ denotes the derivative under the identification (2.2) for a family of operators acting on $L^2(K, H_\sigma)$. Now we can get the invariant part of $W_\chi(h)$ by subtracting the non-invariant part as follows,

$$\begin{aligned} I_P(h) = & \int_N \int_K h(knk^{-1}) \log|\log n|\, dk\, dn \\ & - \frac{1}{2}\left(\frac{1}{2\pi} \mathrm{p.v.} \sum_{\sigma \in \hat{M}} d(\sigma) \int_{-\infty}^{\infty} J_P(\sigma, \lambda : h)\, d\lambda - \sum_{\sigma \in \hat{M}} d(\sigma) \frac{n(\sigma)}{2} \Theta_{\sigma,0}(h) \right) \end{aligned} \tag{5.2}$$

where $2n(\sigma)$ is the order of the zero of $p(\sigma,\lambda)$ at $\lambda=0$. Now one can consider the Fourier transform of invariant tempered distribution I_P for $h\in\mathcal{C}^2(G)$, which is expressed in terms of the discrete series and the principal series.

Let $H_\alpha\in\mathfrak{t}_{\mathbb{C}}$ be the coroot corresponding to $\alpha\in\pm\Sigma_G$, that is, $\alpha(H_\alpha)=2, \alpha'(H_\alpha)\in\mathbb{Z}$ for all $\alpha,\alpha'\in\pm\Sigma_G$, and let

$$\Pi=\prod_{\alpha\in\Sigma_M} H_\alpha, \tag{5.3}$$

which is an element of the symmetric algebra $S(\mathfrak{t}_{\mathfrak{m}\mathbb{C}})$. We denote the simple reflection corresponding to α by s_α for $\alpha\in\Sigma_G$. By Corollary on pp. 96 of [13] (taking $\lambda_P=\frac{\beta}{2}$ with $\beta(H_\beta)=2$), we have

Proposition 5.1. *For $h\in\mathcal{C}^2(G)-\mathcal{C}^2_0(G)$ where $\mathcal{C}^2_0(G)$ is the subspace of the cusp form in $\mathcal{C}^2(G)$,*

$$I_P(h)=\frac{1}{2}\cdot\frac{1}{2\pi}\sum_{\sigma\in\hat{M}}\int_{-\infty}^{\infty}\Omega(\sigma,-\lambda)\Theta_{\sigma,\lambda}(h)\ d\lambda \tag{5.4}$$

where

$$\Omega(\sigma,\lambda)=2d(\sigma)\psi(1)-\frac{1}{2}\sum_{\alpha\in\Sigma_A}\beta(H_\alpha)\frac{\Pi(s_\alpha\lambda_\sigma)}{\Pi(\rho_M)}\Big(\psi(1+\lambda_\sigma(H_\alpha))+\psi(1-\lambda_\sigma(H_\alpha))\Big). \tag{5.5}$$

Here ψ is the digamma function and $\lambda_\sigma-\rho_M$ is the highest weight of $(\sigma,i\lambda)\in\widehat{M}\times i\mathfrak{a}$.

Remark 5.2. By the lemma 5 in [3], if $G=\mathrm{SO}_0(d,1)$ for $d\geq 3$, the equality (5.4) still holds without any contribution from the discrete series for any $h\in\mathcal{C}^2(G)$.

5.2. **Computation for σ_k.** To express $W_\chi(h)$ in terms of the elements in $\widehat{G}$, we use (5.2) and (5.5).

First let us investigate the last term on the right side of (5.2). From (4.3), we have

$$\begin{aligned} n(\sigma_k)=&1 \qquad (0\leq k\leq n-1), \qquad n(\sigma_n^{\pm})=0 \qquad \text{if}\quad d=2n+1,\\ n(\sigma_k)=&1 \qquad (0\leq k\leq n-1) \qquad\qquad\qquad\qquad \text{if}\quad d=2n. \end{aligned}$$

Next we consider the term given by $J_P(\sigma,\lambda:h)$ in (5.2). For a fixed irreducible representation τ, it is known that the Harish-Chandra C-function $C_\tau(\sigma,i\lambda)$ satisfies

$$T_\tau J_{\bar{P}|P}(\sigma,\lambda)^{-1}\partial_{i\lambda}J_{\bar{P}|P}(\sigma,\lambda) \ = \ C_\tau(\sigma,i\lambda)^{-1}\partial_{i\lambda}C_\tau(\sigma,i\lambda)T_\tau$$

where T_τ is the projection to τ-isotypic component of $\mathcal{H}_{\sigma,\lambda}$. Hence, if h is of τ-type, we have

$$J_P(\sigma,\lambda:h) \ = \ -\Theta_{\sigma,\lambda}(h)\,C_\tau(\sigma,i\lambda)^{-1}\partial_{i\lambda}C_\tau(\sigma,i\lambda) \tag{5.6}$$

when $[\tau|_M:\sigma]\neq 0$. By Theorem 8.2 in [4], we can derive the following equalities:

(1) When $d=2n+1$,

$$\begin{aligned} \partial_{i\lambda}\log C_{\tau_k}(\sigma_k,i\lambda) \ &= \ \frac{1}{i\lambda+n-k}-\left(\frac{1}{i\lambda}+\cdots+\frac{1}{i\lambda+n}\right),\\ \partial_{i\lambda}\log C_{\tau_k}(\sigma_{k-1},i\lambda) \ &= \ \frac{1}{i\lambda-n+k-1}-\left(\frac{1}{i\lambda}+\cdots+\frac{1}{i\lambda+n}\right) \end{aligned} \tag{5.7}$$

where σ_n means $\sigma_n^{\pm}$.

(2) When $d = 2n$,

$$\partial_{i\lambda} \log C_{\tau_k}(\sigma_k, i\lambda) = \frac{1}{i\lambda + n - k - \frac{1}{2}} + \Big(\psi(i\lambda) - \psi(i\lambda + n + \frac{1}{2}) \Big) + 2\log 2, \tag{5.8}$$

$$\partial_{i\lambda} \log C_{\tau_k}(\sigma_{k-1}, i\lambda) = \frac{1}{i\lambda - n + k - \frac{1}{2}} + \Big(\psi(i\lambda) - \psi(i\lambda + n + \frac{1}{2}) \Big) + 2\log 2$$

where τ_n means $\tau_n^{\pm}$.

Now the remaining task to compute $W_\chi(h)$ is to obtain an explicit form of $\Omega(\sigma_k, \lambda)$ which express $I_P(h)$ in terms of the principal series.

Theorem 5.3. *For the representations σ_k of $\mathrm{SO}(d-1)$ for $0 \le k \le [\frac{d-1}{2}]$, we have*

$$\Omega(\sigma_k, \lambda) = -\frac{d(\sigma_k)}{2} \Big(\psi(i\lambda - \frac{d-1}{2}) + \psi(-i\lambda - \frac{d-1}{2}) + \psi(i\lambda + 1) + \psi(-i\lambda + 1) \Big)$$
$$+ \frac{(-1)^k (\frac{d-1}{2} - k)}{\lambda^2 + (\frac{d-1}{2} - k)^2} \Big(\sum_{j=0}^{k-1} (-1)^j d(\sigma_j) + \sum_{j=k+1}^{d-1} (-1)^{j+1} d(\sigma_j) \Big) - P_k^d(\lambda)$$

where σ_n denotes $\sigma_n^{\pm}$ if $d = 2n+1$ and $P_k^d(\lambda)$ is an even polynomial of $2n-4$ degree for $d = 2n+1 \ge 5$ or $d = 2n \ge 4$ and a constant for $d = 3, 2$.

The proof of this theorem will be given in the Appendix A.

For τ_k, σ_ℓ with $[\tau_k|_M : \sigma_\ell] \ne 0$, we put

$$\Phi(\tau_k, \sigma_\ell, \lambda) := -d(\sigma_\ell) \frac{1}{2} \big(\partial_{i\lambda} \log C_{\tau_k}(\sigma_\ell, i\lambda) - \partial_{i\lambda} \log C_{\tau_k}(\sigma_\ell, -i\lambda) \big).$$

Then by the equalities (5.7), (5.8) and Theorem 5.3, denoting a possibly different polynomial by the same notation $P_k^d(\lambda)$ (the change happens in the constant term of $P_k^d(\lambda)$ only when $d = 2n$) we have

Corollary 5.4. *The following equalities hold,*

$$\Omega(\sigma_k, \lambda) + \Phi(\tau_k, \sigma_k, \lambda) = -d(\sigma_k) \big(\psi(i\lambda + 1) + \psi(-i\lambda + 1) \big)$$
$$+ \frac{(-1)^k (\frac{d-1}{2} - k)}{\lambda^2 + (\frac{d-1}{2} - k)^2} \Big(\sum_{j=0}^{k-1} (-1)^j d(\sigma_j) + \sum_{j=k}^{d-1} (-1)^{j+1} d(\sigma_j) \Big) - P_k^d(\lambda),$$

$$\Omega(\sigma_k, \lambda) + \Phi(\tau_{k+1}, \sigma_k, \lambda) = -d(\sigma_k) \big(\psi(i\lambda + 1) + \psi(-i\lambda + 1) \big)$$
$$+ \frac{(-1)^k (\frac{d-1}{2} - k)}{\lambda^2 + (\frac{d-1}{2} - k)^2} \Big(\sum_{j=0}^{k} (-1)^j d(\sigma_j) + \sum_{j=k+1}^{d-1} (-1)^{j+1} d(\sigma_j) \Big) - P_k^d(\lambda).$$

6. Zeta regularized determinant for hyperbolic manifolds with cusps

Now let us recall that the heat operator $e^{-t\Delta_k}$ over X_Γ is not of trace class, so that we can not take its usual trace. To overcome this, we follow the idea of Melrose in [16] as follows. If the heat operator $e^{-t\Delta_k}$ would be of trace class, then its trace is the same as $\int_{X_\Gamma} \mathrm{tr}\, \big(e^{-t\Delta_k}(x,x) \big)\, dx$, although this integral diverges in our case. However, we could remove the diverging part of the expansion of

$$\int_{X_\Gamma(u)} \mathrm{tr}\, \big(e^{-t\Delta_k}(x,x) \big)\, dx \qquad \text{as} \qquad u \to \infty$$

by Theorem 3.1 and define the regularized trace $\mathrm{Tr}_{\mathrm{r}}(\cdot)$ of $e^{-t\Delta_k}$ to be the remaining finite part of it. Then we have

$$\begin{aligned}\mathrm{Tr}_{\mathrm{r}}\left(e^{-t\Delta_k}\right) = &\sum_{\lambda_j\in\sigma_p(\Delta_k)} e^{-t\lambda_j} + \sum_{[\tau_k|_M:\sigma_\ell]\neq 0}\Big(\frac{d(\sigma_\ell)}{4}e^{-td_\ell^2}\,\mathrm{tr}\left(C^k_\chi(\sigma_\ell,0)\right)\\ &- \frac{d(\sigma_\ell)}{4\pi}\int_{-\infty}^{\infty} e^{-t(\lambda^2+d_\ell^2)}\mathrm{tr}\big(C^k_\chi(\sigma_\ell,-i\lambda)\partial_{i\lambda}C^k_\chi(\sigma_\ell,i\lambda)\big)\,d\lambda\Big)\end{aligned} \tag{6.1}$$

where $\sigma_p(\Delta_k)$ denotes the point spectrum of Δ_k, $d_\ell = (\frac{d-1}{2}-\ell)$, $d(\sigma_\ell)=\dim(V_{\sigma_\ell})$. Let us observe that the right-hand side of (6.1) is the same as the geometric side of the Selberg trace formula applied to the test function h^k_t over G, so that

$$\mathrm{Tr}_{\mathrm{r}}\left(e^{-t\Delta_k}\right) = I_\chi(h^k_t) + H_\chi(h^k_t) + U_\chi(h^k_t) + W_\chi(h^k_t) \tag{6.2}$$

by (4.1). Using this regularized trace, let us define the spectral zeta function of Δ_k by

$$\zeta_{\Delta_k}(s) := \frac{1}{\Gamma(s)}\left(\int_0^1 + \int_1^\infty\right) t^{s-1}\mathrm{Tr}_{\mathrm{r}}\left(e^{-t\Delta_k} - P_k\right)\,dt \tag{6.3}$$

where P_k denotes the orthogonal projection onto $\ker_{L^2}(\Delta_k)$. Here the small and large time integrals $\int_0^1$, $\int_1^\infty$ are defined for $\mathrm{Re}(s)\gg 0$ and $\mathrm{Re}(s)\ll 0$ respectively. This decomposition of the integral over the small and large times is needed when $d=2n+1$, $k=n$ since the continuous spectrum of Δ_k reaches zero, that is, the heat operator $e^{-t\Delta_k}$ does not decay exponentially as t $\to\infty$. To state the theorem on the meromorphic extension of $\zeta_{\Delta_k}(s)$, we introduce a notation

$$\mathrm{d}(d,k) := \Big((-1)^{k-1}\big(\sum_{j=0}^{k-1}(-1)^j d(\sigma_j)\big)\Big) = \binom{d-1}{k} - \binom{d-2}{k}.$$

Then we have

$$-2\,\mathrm{d}(d,k) = -2\Big((-1)^{k-1}\big(\sum_{j=0}^{k-1}(-1)^j d(\sigma_j)\big)\Big) = (-1)^k\left(\sum_{j=0}^{k-1}(-1)^j d(\sigma_j) + \sum_{j=k}^{d-1}(-1)^{j+1}d(\sigma_j)\right),$$

which appeared in Corollary 5.4. Now we have

Theorem 6.1. *The spectral zeta function $\zeta_{\Delta_k}(s)=\zeta_{\Delta_{d-k}}(s)$ has a meromorphic extension over $\mathbb{C}$, which has the following form if $d=2n$*

$$\Gamma(s)\zeta_{\Delta_k}(s) = \sum_{j=-n}^{\infty}\frac{a_j}{s+j} + \sum_{j=-(n-2)}^{\infty}\frac{a'_j}{s+j-\frac12} + \sum_{j=0}^{\infty}\frac{b_j}{(s+j-\frac12)^2} - \frac{\beta_k+\delta_n(k)\eta^e}{s} + H_e(s),$$

and if $d=2n+1$

$$\Gamma(s)\zeta_{\Delta_k}(s) = \sum_{j=-n}^{\infty}\frac{a_j}{s+j-\frac12} + \sum_{j=0}^{\infty}\frac{b_j}{(s+j-\frac12)^2} + \delta_n(k)\sum_{j=0}^{\infty}\frac{c_j}{s-j-\frac12} - \frac{\beta_k+\delta_n(k)\eta^o}{s} + H_o(s)$$

for some constants a_j, a'_j, b_j, c_j where $\beta_k=\dim\ker_{L^2}(\Delta_k)$, $\delta_n(k)$ equals 1 if $k=n$ or $k=(d-n)$ and vanishes otherwise,

$$\eta^e := d_c(\chi)\mathrm{d}(d,n), \qquad \eta^o := -\frac{1}{2}(d_c(\chi)\mathrm{d}(d,n-1))$$

and $H_e(s)$, $H_o(s)$ are entire functions. In particular, $\zeta_{\Delta_k}(s)$ is regular at $s=0$.

Proof. Let us first deal with the large time contribution $\int_1^\infty \cdot\, dt$ in (6.3). The continuous spectrum of Δ_k is given by the union of the half intervals $[(\frac{d-1}{2}-\ell)^2,\infty)$ for $\ell = k, k-1$, hence the bottom of the continuous spectrum of Δ_k does not reach zero unless $d = 2n+1$ and $\ell = k = n$. Equivalently, $\mathrm{Tr}_{\mathrm{r}}\left(e^{-t\Delta_k}\right)$ decays exponentially as $t \to \infty$ for other cases, which we can see easily from the right-hand side of (6.1). Therefore, the large time contribution to the meromorphic extension is trivial unless $d = 2n + 1$ and $\ell = k = n$. Now for this case, we observe the following expansion at $\lambda = 0$,

$$\mathrm{tr}\big(C^n_\chi(\sigma_n, -i\lambda)\partial_{i\lambda} C^n_\chi(\sigma_n, i\lambda)\big) = \sum_{j=0}^{\infty} a_{2j}\lambda^{2j},$$

which follows from (3.3). From this, we see that

$$\int_{-1}^{1} e^{-t\lambda^2}\mathrm{tr}\big(C^n_\chi(\sigma_n, -i\lambda)\partial_{i\lambda} C^n_\chi(\sigma_n, i\lambda)\big)\, d\lambda \quad\sim\quad \sum_{j=0}^{\infty} b_j t^{-(j+\frac{1}{2})} \quad \text{as} \quad t\to\infty. \tag{6.4}$$

The corresponding integrals over $(-\infty,1]_\lambda \cup [1,\infty)_\lambda$ decay exponentially as $t \to \infty$. For $d = 2n + 1$, σ_n is un-ramified so that $\mathrm{tr}\left(C^n_\chi(\sigma_n, 0)\right) = 0$. Hence the residual term vanishes for this case. Now the expansion (6.4) and these facts imply that the large time integral $\int_1^\infty$ is well defined for $\mathrm{Re}(s) < \frac{1}{2}$ and extends meromorphically to the whole complex plane with the following form

$$\int_1^\infty t^{s-1}\mathrm{Tr}_{\mathrm{r}}\left(e^{-t\Delta_n} - P_n\right)\, dt = \sum_{j=0}^{\infty} \frac{c_j}{s-j-\frac{1}{2}} + H_1(s) \tag{6.5}$$

for some constants c_j and a holomorphic function $H_1(s)$.

Next to deal with the small time integral $\int_0^1 \cdot\, dt$, we use the right-hand side of the equality (6.2). For $I_\chi(h^k_t)$, we separate the cases $d = 2n$ and $d = 2n + 1$. First, if $d = 2n$ recall that the Plancherel measure $P(\sigma_\ell, \lambda)$ is a sum of $\lambda^{2k+1}\tanh(\pi\lambda)$ with $0 \le k \le n-1$ from (4.3). Now we observe

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-t\lambda^2}\lambda^{2k+1}\tanh(\pi\lambda)\, d\lambda =&(-1)^k\partial_t^k \int_{-\infty}^{\infty} e^{-t\lambda^2}\lambda\tanh(\pi\lambda)\, d\lambda \\ =&(-1)^k\partial_t^k \int_0^\infty e^{-tx}\tanh(\pi\sqrt{x})\, dx \\ =&(-1)^k\partial_t^k \Big(t^{-1}\frac{\pi}{2}\int_0^\infty e^{-tx}(\frac{\cosh^{-2}(\pi\sqrt{x})}{\sqrt{x}})\, dx\Big) \\ =&(-1)^k\partial_t^k \Big(t^{-1}\frac{\pi}{2}\int_0^\infty \sum_{j=0}^{\infty}\frac{(-tx)^j}{j!}(\frac{\cosh^{-2}(\pi\sqrt{x})}{\sqrt{x}})\, dx\Big).\end{aligned} \tag{6.6}$$

Hence if $d = 2n$, we conclude

$$I_\chi(h^k_t) = \sum_{\ell=k,k-1} e^{-td_\ell^2}\cdot \sum_{j=0}^{\infty} a_{\ell j} t^{-n+j} + \tilde{\delta}_n(k)\tilde{c}(\Gamma,\chi,\tau_n) \tag{6.7}$$

where $a_{\ell j}$ are some constants and $\tilde{\delta}_n(k)\tilde{c}(\Gamma,\chi,\tau_n)$ is the contribution from the first term on the right hand side of (4.2). Second, if $d = 2n+1$ recall that the Plancherel measure $p(\sigma_\ell, \lambda)$

is a polynomial of order $2n$ from (4.3). Hence we can easily see

$$I_\chi(h_t^k) = \sum_{\ell=k,k-1} e^{-td_\ell^2} \cdot \sum_{j=0}^{n} a_{\ell j} t^{-n+j-\frac{1}{2}} \tag{6.8}$$

for some constants $a_{\ell j}$. For $H_\Gamma(h_t^k)$, by (4.6), we have

$$H_\chi(h_t^k) \ \sim\ a\, e^{-\frac{c^2}{4t}} \quad \text{as} \quad t \to 0 \tag{6.9}$$

for a constant a and $c := \min_{\{\gamma:\text{hyperbolic}\}} l(C_\gamma)$ is a positive real number. For $U_\chi(h_t^k)$, by (4.14) and (4.15) we can see that this can be dealt as $I_\chi(h_t^k)$ for $d = 2n+1$ and it consists of the terms with $j = n$ in (6.8). For $W_\chi(h_t^k)$, by Corollary 5.4, we have

$$W_\chi(h_t^k) = \frac{d_c(\chi)}{4\pi} \sum_{\ell=k,k-1} \int_{-\infty}^{\infty} e^{-t(\lambda^2+d_\ell^2)} \left(P_\ell(\lambda) + Q_\ell(\lambda) + R_\ell^k(\lambda)\right) d\lambda. \tag{6.10}$$

Here $P_\ell(\lambda)$ is an even polynomial of degree at most $(2n-4)$ for $d = 2n+1$ or $d = 2n$,

$$Q_\ell(\lambda) = -d(\sigma_\ell)\Big(\psi(i\lambda+1) + \psi(-i\lambda+1)\Big), \qquad R_\ell^k(\lambda) = -(-1)^{k-\ell} 2\,\mathrm{d}(d,k) \frac{d_\ell}{\lambda^2 + d_\ell^2}.$$

It is easy to see that the contribution of $P_\ell(\lambda)$ is just the same as (6.8) replacing n by $(n-2)$. For $Q_\ell(\lambda)$, we use the following asymptotic expansion

$$\psi(z+1) \ \sim\ \log z + \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k) z^{2k}} \qquad \text{as} \quad z \to \infty$$

where the B_{2k}'s are Bernoulli numbers, to obtain

$$\int_{-\infty}^{\infty} e^{-t\lambda^2} Q_\ell(\lambda)\, d\lambda \ \sim\ \sum_{j=0}^{\infty} a_j t^{j-\frac{1}{2}} + b_0 t^{-\frac{1}{2}} \log t \qquad \text{as} \quad t \to 0 \tag{6.11}$$

for some constants a_j and b_0. By an elementary computation,

$$\frac{d}{dt} \int_{-\infty}^{\infty} e^{-t(\lambda^2+d_\ell^2)} \frac{d_\ell}{\lambda^2 + d_\ell^2}\, d\lambda = -d_\ell \frac{\sqrt{\pi}}{\sqrt{t}} e^{-td_\ell^2}, \tag{6.12}$$

which implies

$$\int_{-\infty}^{\infty} e^{-t(\lambda^2+d_\ell^2)} \frac{d_\ell}{\lambda^2 + d_\ell^2}\, d\lambda = \pi + \sum_{j=0}^{\infty} a_j t^{j+\frac{1}{2}} \tag{6.13}$$

for some constants a_j. By (6.8), (6.9), (6.11), (6.13), and the Taylor expansion of $e^{-td_\ell^2}$ at $t = 0$, if $d = 2n$

$$\begin{aligned} I_\chi(h_t^k) + H_\chi(h_t^k) &+ U_\chi(h_t^k) + W_\chi(h_t^k) \\ &\sim\ -\delta_n(k)\eta^e + \sum_{j=-n}^{\infty} a_j t^j + \sum_{j=-(n-2)}^{\infty} a'_j t^{j-\frac{1}{2}} + \sum_{j=0}^{\infty} b_j t^{j-\frac{1}{2}} \log t \qquad \text{as} \quad t \to 0 \end{aligned}$$

for some constants a_j, a'_j, b_j where $\eta^e = d_c(\chi)\mathrm{d}(d,n)$, and if $d = 2n+1$

$$I_\chi(h_t^k) + H_\chi(h_t^k) + U_\chi(h_t^k) + W_\chi(h_t^k) \ \sim\ -\delta_n(k)\eta^o + \sum_{j=-n}^{\infty} a_j t^{j-\frac{1}{2}} + \sum_{j=0}^{\infty} b_j t^{j-\frac{1}{2}} \log t \qquad \text{as} \quad t \to 0$$

for constants a_j, b_j where $\eta^o = -\frac{1}{2}(d_c(\chi)\mathrm{d}(d, n-1))$. Therefore the small time integral $\int_0^1$ is well defined for $\mathrm{Re}(s) > \frac{d}{2}$ and extends meromorphically on $\mathbb{C}$ with the following form if $d = 2n$

$$\Gamma(s)\zeta_{\Delta_k}(s) = \sum_{j=-n}^{\infty} \frac{a_j}{s+j} + \sum_{j=-(n-2)}^{\infty} \frac{a'_j}{s+j-\frac{1}{2}} + \sum_{j=0}^{\infty} \frac{b_j}{(s+j-\frac{1}{2})^2} - \frac{\beta_k + \delta_n(k)\eta^e}{s} + H_2(s),$$

and if $d = 2n+1$

$$\Gamma(s)\zeta_{\Delta_k}(s) = \sum_{j=-n}^{\infty} \frac{a_j}{s+j-\frac{1}{2}} + \sum_{j=0}^{\infty} \frac{b_j}{(s+j-\frac{1}{2})^2} - \frac{\beta_k + \delta_n(k)\eta^o}{s} + H_2(s)$$

for some (new) constants a_j, a'_j, b_j and a holomorphic function $H_2(z)$. For $d = 2n+1$ and $k = n$, combining this and (6.5) completes the proof.

□

By Theorem 6.1, we can define the regularized determinant of Δ_k by

$$\det{}_\zeta \Delta_k := \exp\left(-\frac{d}{ds}\Big|_{s=0} \zeta_{\Delta_k}(s) \right)$$

and the analytic torsion $T(X_\Gamma, \rho)$ by

$$T(X_\Gamma, \chi) := \frac{\det_\zeta \Delta_1}{(\det_\zeta \Delta_2)^2} \cdot \frac{(\det_\zeta \Delta_3)^3}{(\det_\zeta \Delta_4)^4} \ldots (\det{}_\zeta \Delta_{d-1})^{(-1)^d (d-1)} \cdot (\det{}_\zeta \Delta_d)^{(-1)^{d+1} d}.$$

Note that our definition of analytic torsion is a generalization of the original one given in [24], which reduces to (the square of) the original one in [24] when X_Γ is compact. We also remark that a similar definition of the analytic torsion was introduced by Hassell in [12] using the b-trace.

For a hyperbolic manifold X_Γ with cusps, $T(X_\Gamma, \chi)$ is nontrivial even if $d = 2n$ as we will see in Section 8.

In the following section, we will relate $T(X_\Gamma, \chi)$ with the leading coefficient $R^*_\chi(0)$ of the Ruelle zeta function $R_\chi(s)$ at $s = 0$. To do this, we will need following expression of $R_\chi(s)$ (for instance, see pp. 532 in [6]),

(6.14) $$R_\chi(s) = \prod_{k=0}^{d-1} Z_\chi(\sigma_k, s+k)^{(-1)^{k+1}}$$

in terms of the Selberg zeta function $Z_\chi(\sigma_\ell, s)$ defined by

(6.15) $$Z_\chi(\sigma_k, s) := \exp\Big(-\sum_{\gamma \in \Gamma_{\mathrm{hyp}}} \mathrm{tr}\, \chi(\gamma) j(\gamma)^{-1} D(\gamma)^{-1} \overline{\mathrm{tr}\, \sigma_k(m_\gamma)} e^{-(s-\frac{d-1}{2}) l(C_\gamma)} \Big)$$

for $\mathrm{Re}(s) > d-1$. Here we may assume that γ is conjugate to $a_\gamma m_\gamma \in A^+M$ and $D(\gamma) = D(a_\gamma m_\gamma)$. We also put $Z_\chi(\sigma_n, s) = Z_\chi(\sigma_n^+, s) \cdot Z_\chi(\sigma_n^-, s)$ when $d = 2n+1$. By Theorem 4.6 in [10], the Selberg zeta function $Z_\chi(\sigma_\ell, s)$ has a meromorphic extension over $\mathbb{C}$.

7. Proof of Theorem 1.1

Throughout this section, we assume $d = 2n + 1$.

First, taking the Mellin transform $M(\cdot)$ of the equality (6.2), we have

$$M\big(\mathrm{Tr}_r(e^{-t\Delta_k}) - \beta_k\big) = MI_\chi(h_t^k) + MH_\chi(h_t^k) + MU_\chi(h_t^k) + MW_\chi(h_t^k) - M(\beta_k). \tag{7.1}$$

For the left-hand side of (7.1), we have

Lemma 7.1. *The following equality holds,*

$$\lim_{s\to 0}\ \Big(M\big(\mathrm{Tr}_r(e^{-t\Delta_k}) - \beta_k\big)(s) + \Gamma(s)\big(\beta_k + \delta_n(k)\eta^o\big)\Big) \ = \ \zeta'_{\Delta_k}(0).$$

Proof. From the definition, we have

$$\zeta'_{\Delta_k}(0) = \lim_{s\to 0} \frac{1}{s}\big(\zeta_{\Delta_k}(s) - \zeta_{\Delta_k}(0)\big) = \lim_{s\to 0}\Big(M\big(\mathrm{Tr}_r(e^{-t\Delta_k}) - \beta_k\big) - \Gamma(s)\zeta_{\Delta_k}(0)\Big).$$

Moreover, by Theorem 6.1,

$$\zeta_{\Delta_k}(0) = -\beta_k - \delta_n(k)\eta^o.$$

Our lemma now follows easily. □

Now we want to obtain the explicit form of each term at $s = 0$ on the right-hand side of (7.1). First, the last term $M(\beta_k)$ is defined by

$$M(\beta_k)(s) = \int_0^1 t^{s-1}\beta_k\, dt + \int_1^\infty t^{s-1}\beta_k\, dt$$

where the first (second) term on the right-hand side has a meromorphic extension from the half plane with $\mathrm{Re}(s) \gg 0$ ($\mathrm{Re}(s) \ll 0$). These terms equal $\frac{\beta_k}{s}$ and $-\frac{\beta_k}{s}$, respectively so that

$$M(\beta_k)(s) \equiv 0. \tag{7.2}$$

The term $MI_\chi(h_t^k)$ is computed in Lemma 3 on pp. 533 in [6]. Let us present some details of these computations for completeness. By Proposition 2.3, (4.2) and (4.3),

$$I_\chi(h_t^k) = \dim V_\chi \cdot \mathrm{vol}(\Gamma\backslash G) \sum_{\ell=k,k-1} e^{-t(n-\ell)^2}\, \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-t\lambda^2} p(\sigma_\ell,\lambda)\, d\lambda$$

where $p(\sigma_\ell,\lambda)$ is an even polynomial. Now for the integral part, each monomial can be treated as

$$\int_{-\infty}^{\infty} e^{-t\lambda^2}\lambda^{2a}\, d\lambda = \big(-\frac{d}{dt}\big)^a \int_{-\infty}^{\infty} e^{-t\lambda^2}\, d\lambda = b_a t^{-a-\frac{1}{2}} \tag{7.3}$$

where $b_a = \sqrt{\pi}\frac{1}{2}\cdot\frac{3}{2}\dots\frac{2a-1}{2}$ so that $MI_\chi(h_t^k)(s)$ consists of

$$E_a(s) := b_a \int_0^\infty t^{s-a-\frac{3}{2}} e^{-t(n-\ell)^2}\, dt = b_a\Gamma(s-a-\frac{1}{2})(n-\ell)^{-2(s-a-\frac{1}{2})} \tag{7.4}$$

for $\mathrm{Re}(s) > a + \frac{1}{2}$. When $\ell < n$, $MI_\chi(h_t^k)$ has a meromorphic extension over $\mathbb{C}$ and

$$\begin{aligned} E_a(0) &= b_a\Gamma(-a-\frac{1}{2})(n-\ell)^{2a+1} \\ &= (-1)^{a+1}\frac{2\pi}{2a+1}(n-\ell)^{2a+1} = -2\pi\int_0^{n-\ell} (i\lambda)^{2a}\, d\lambda. \end{aligned}$$

When $\ell = n$, we split the integral defining $E_a(s)$ as we did for $M(\beta_k)$,

$$E_a(s) = b_a \int_0^1 t^{s-a-\frac{3}{2}}\,dt + b_a \int_1^\infty t^{s-a-\frac{3}{2}}\,dt$$

where the first (second) term on the right-hand side has a meromorphic extension from the half plane with $\mathrm{Re}(s) \gg 0$ ($\mathrm{Re}(s) \ll 0$). These terms equal $\frac{b_a}{s-a-\frac{1}{2}}$ and $-\frac{b_a}{s-a-\frac{1}{2}}$, respectively, so that

$$E_a(s) \equiv 0 \qquad \text{if} \quad \ell = n.$$

In conclusion,

$$MI_\chi(h_t^k)(0) = -\frac{1}{2}\,\mathrm{dim}V_\chi \cdot \mathrm{vol}(\Gamma\backslash G) \sum_{\ell=k,k-1} \int_0^{n-\ell} p(\sigma_\ell, i\lambda)\,d\lambda.$$

The term $U_\chi(h_t^k)$ and the part with $P_\ell(\lambda)$ in (6.10) denoted by $W_\chi^1(h_t^k)$ can be dealt in the same way as we did for $I_\chi(h_t^k)$ and we have

$$M\big(I_\chi(h_t^k) + U_\chi(h_t^k) + W_\chi^1(h_t^k)\big)(0) = -\frac{1}{2} \sum_{\ell=k,k-1} \int_0^{n-\ell} \widetilde{P}_\ell(i\lambda)\,d\lambda. \tag{7.5}$$

Here

$$\widetilde{P}_\ell(s) = \mathrm{dim}V_\chi \cdot \mathrm{vol}(\Gamma\backslash G) p(\sigma_\ell, s) - d_c(\chi) P_\ell^d(s) + C(\chi, k) \tag{7.6}$$

where $C(\chi, k)$ is a constant from $U_\chi(h_t^k)$, which is determined by (4.14), (4.15).

For $MH_\chi(h_t^k)$, first we recall

$$t^{s-1} = \frac{1}{\Gamma(1-s)} \int_0^\infty (x(x+2c))^{-s} e^{-x(x+2c)t}(2x+2c)\,dx \qquad \text{for} \quad \mathrm{Re}(s) < 0$$

where $c > 0$ following [18], [6]. Now using

$$\int_0^\infty e^{-x(x+2c)t}(2x+2c)\frac{1}{\sqrt{4\pi t}} e^{-\frac{l^2}{4t}} e^{-tc^2}\,dt = e^{-l(x+c)}$$

and putting $c = (n-\ell)$, $l = l(C_\gamma)$ in (4.6), we have

$$MH_\chi(h_t^k)(s) = \sum_{\ell=k,k-1} \frac{1}{\Gamma(1-s)} \int_0^\infty (x(x+2(n-\ell)))^{-s} \big(\frac{d}{dx} \log Z_\chi(\sigma_\ell, 2n-\ell+x)\big)\,dx \tag{7.7}$$

for $\mathrm{Re}(s) < 0$. By Theorem 4.6 in [10], the Selberg zeta function $Z_\chi(\sigma_\ell, s)$ has a meromorphic extension over $\mathbb{C}$. In particular, it follows that $Z_\chi(\sigma_\ell, 2n-\ell+x)$ has the following form near $x = 0$,

$$Z_\chi(\sigma_\ell, 2n-\ell+x) = Z_{2n-\ell}\, x^{-r_{2n-\ell}}(1 + O(x)) \tag{7.8}$$

where $r_{2n-\ell}$ denotes the order of singularity of $Z_\chi(\sigma_\ell, s)$ at $s = 2n-\ell$. By Theorem 2.1 of [9] or Theorem 4.6 of [10],

$$r_{2n-\ell} = \begin{cases} -\alpha_\ell & \text{if } \ell \neq n \\ -2\alpha_n & \text{if } \ell = n \end{cases} \tag{7.9}$$

where

$$\alpha_k := \beta_k - \beta_{k-1} + \beta_{k-2} - \ldots \pm \beta_0.$$

Using this, the integral part on the right-hand side of (7.7) can be analyzed as

$$\begin{aligned}(7.10)\quad &\int_0^\infty (x(x+2(n-\ell)))^{-s}\big(\frac{d}{dx}\log Z_\chi(\sigma_\ell, 2n-\ell+x)\big)\,dx\\ &=\int_\epsilon^\infty \frac{d}{dx}\log Z_\chi(\sigma_\ell,2n-\ell+x)\,dx + O(s)+O(\epsilon) - r_{2n-\ell}\int_0^\epsilon x^{-s-1}(x+2(n-\ell))^{-s}\,dx\end{aligned}$$

where

$$\begin{aligned}(7.11)\quad &\int_0^\epsilon x^{-s-1}(x+2(n-\ell))^{-s}\,dx\\ &=\begin{cases}\log(2(n-\ell)\epsilon)-\frac{1}{s}+O(s)+O(\epsilon^{1-|s|}) & \text{if}\quad \ell<n,\\ \log\epsilon-\frac{1}{2s}+O(s) & \text{if}\quad \ell=n\end{cases}.\end{aligned}$$

From (7.7), (7.9), (7.10) and (7.11), for small $s<0$,

$$MH_\chi(h_t^k)(s) = \sum_{\ell=k,k-1}\big(-\log Z_{2n-\ell} - r_{2n-\ell}\log(2(n-\ell)) - \frac{\alpha_\ell}{s}\big) + O(s) + O(\epsilon^{\frac{1}{2}}).$$

Hence we obtain

$$(7.12)\quad \lim_{s\to 0}\big(MH_\chi(h_t^k)(s) - \beta_k\Gamma(s)\big) = \sum_{\ell=k,k-1} -\log Z_{2n-\ell} + \alpha_\ell(\log 2(n-\ell))$$

where the term $\log(2(n-\ell))$ disappears when $\ell=n$.

To analyze the remaining terms of $W_\chi(h_t^k)$ in (6.10), let us denote the corresponding parts of $W_\chi(h_t^k)$ with $Q_\ell(\lambda)$, $R_\ell(\lambda)$ for $\ell=k,k-1$ by $W_\chi^2(h_t^k)$, $W_\chi^3(h_t^k)$ respectively, that is,

$$\begin{aligned}W_\chi^2(h_t^k) &= -\frac{d_c(\chi)}{4\pi}\sum_{\ell=k,k-1} d(\sigma_\ell)\int_{-\infty}^\infty e^{-t(\lambda^2+(n-\ell)^2)}\big(\psi(i\lambda+1)+\psi(-i\lambda+1)\big)\,d\lambda,\\ W_\chi^3(h_t^k) &= -\frac{d_c(\chi)\mathrm{d}(d,k)}{2\pi}\sum_{\ell=k,k-1}(-1)^{k-\ell}\int_{-\infty}^\infty e^{-t(\lambda^2+(n-\ell)^2)}\frac{(n-\ell)}{\lambda^2+(n-\ell)^2}\,d\lambda.\end{aligned}$$

Now, we deal with $W_\chi^2(h_t^k)(0)$.

Lemma 7.2. *The following equality holds,*

$$MW_\chi^2(h_t^k)(0) = d_c(\chi)\sum_{\ell=k,k-1} d(\sigma_\ell)\big(\log\Gamma(n-\ell+1)+C\big)$$

where C is a constant which does not depend on ℓ.

Proof. For $c\in\mathbb{R}$ and $\mathrm{Re}(s)\gg 0$, let

$$f_c(s) = \int_0^\infty t^{s-1}\int_{-\infty}^\infty e^{-t(\lambda^2+c^2)}\big(\psi(i\lambda+1)+\psi(-i\lambda+1)\big)\,d\lambda\,dt.$$

It can be shown that the c-family of functions $f_c(s)$ extends meromorphically over $\mathbb{C}$ and that $f_c(s)$ is regular over $\mathbb{C}-\{\frac{1}{2},-\frac{1}{2},-\frac{3}{2},\ldots\}$ in the same way as the proof of Theorem 6.1. Denoting by $f_c'(s)$ the derivative of $f_c(s)$ with respect to c, for $\mathrm{Re}(s)\gg 0$,

$$f_c'(s) = -2c\int_0^\infty t^s\int_{-\infty}^\infty e^{-t(\lambda^2+c^2)}\big(\psi(i\lambda+1)+\psi(-i\lambda+1)\big)\,d\lambda\,dt = -2cf(s+1),$$

which also holds over $\mathbb{C}-\{\frac{1}{2},-\frac{1}{2},-\frac{3}{2},\dots\}$ by the meromorphic extension. In particular, $f_c(0)$ is smooth for $c\in\mathbb{R}$, and

$$\begin{aligned} f_c'(0) =& -2c\int_0^\infty\int_{-\infty}^\infty e^{-t(\lambda^2+c^2)}\big(\psi(i\lambda+1)+\psi(-i\lambda+1)\big)\,d\lambda\,dt\\ =& -2c\int_{-\infty}^\infty \frac{1}{\lambda^2+c^2}\big(\psi(i\lambda+1)+\psi(-i\lambda+1)\big)\,d\lambda\\ =& -2i\int_{-\infty}^\infty \big(\frac{1}{\lambda+ic}-\frac{1}{\lambda-ic}\big)\psi(i\lambda+1)\,d\lambda=-4\pi\psi(1+c). \end{aligned}$$

From this formula, we see that $f_c(0)=-4\pi\log\Gamma(1+c)+a$ for a constant a for $c\in[0,\infty)$. Applying this to the formula of $W^2_\chi(h^k_t)$, we obtain the expected equality. ☐

Next, for $W^3_\chi(h^k_t)(0)$ we have

Lemma 7.3. *The following equality holds*

$$\lim_{s\to0}\big(MW^3_\chi(h^k_t)(s)+\delta_n(k)\frac{\eta^o}{s}\big)=d_c(\chi)\mathrm{d}(d,k)(\log(n-k)-\log(n-k+1))$$

where the right-hand side is trivial if $k=n$.

Proof. For $c\in(0,\infty)$, we put

$$F(t)=\int_{-\infty}^\infty e^{-t(\lambda^2+c^2)}\frac{c}{\lambda^2+c^2}\,d\lambda.$$

By (6.12), we have

$$\frac{d}{dt}F(t)=-c\frac{\sqrt{\pi}}{\sqrt{t}}e^{-tc^2}.$$

Hence, for $\mathrm{Re}(s)\gg0$,

$$\begin{aligned}\int_0^\infty t^{s-1}F(t)\,dt=&-\frac{1}{s}\int_0^\infty t^s\big(-c\frac{\sqrt{\pi}}{\sqrt{t}}e^{-tc^2}\big)\,dt\\ =&\frac{c\sqrt{\pi}}{s}\int_0^\infty t^{s-\frac12}e^{-tc^2}\,dt=\frac{\sqrt{\pi}}{s}\,c^{-2s}\,\Gamma(s+\frac12).\end{aligned}$$

This implies

$$\begin{aligned}&MW^3_\chi(h^k_t)(s)\\ =&-\frac{d_c(\chi)\mathrm{d}(d,k)}{2\pi}\sum_{\ell=k,k-1}(-1)^{k-\ell}\frac{\sqrt{\pi}}{s}(1-2\log(n-\ell)s+\mathrm{O}(s^2))(\sqrt{\pi}+\Gamma'(1/2)s+\mathrm{O}(s))\\ =&-\frac{d_c(\chi)\mathrm{d}(d,k)}{2\pi}\sum_{\ell=k,k-1}(-1)^{k-\ell}\big(\frac{\pi}{s}-2\pi\log(n-\ell)+\pi\psi(1/2)+\mathrm{O}(s)\big),\end{aligned}$$

which completes the proof. ☐

Combining (7.5), (7.12) and Lemma 7.1, 7.2, and 7.3,

$$\begin{aligned}\det{}_\zeta\Delta_k=&\prod_{\ell=k,k-1}Z_{2n-\ell}\,(2(n-\ell))^{-\alpha_\ell}\exp\big(\frac12\int_0^{n-\ell}\widetilde{P}_\ell(i\lambda)\,d\lambda\big)\\ &\cdot(n-\ell)^{(-1)^{k-\ell+1}d_c(\chi)\mathrm{d}(d,k)}\Gamma(n-\ell+1)^{-d_c(\chi)d(\sigma_\ell)}e^{-d_c(\chi)d(\sigma_\ell)C}\end{aligned}\tag{7.13}$$

where the terms $(2(n-\ell))^{-\alpha_\ell}$ disappear if $\ell = n$. Let us remark that the order $(-1)^{k-\ell+1}$ of $(n-\ell)$ depends on both k and ℓ and this is due to the non-invariant property of the weighted orbital invariant.

From Theorem 2.2 of [9] or Theorem 4.14 of [10], we have

Proposition 7.4. *For $s \in \mathbb{C}$, the following equalities hold*

$$\begin{aligned} &Z_\chi(\sigma_k, s+k)\Gamma(s-n+k+1)^{-d_c(\chi)\, d(\sigma_k)} s^{-d_c(\chi)\, \mathrm{d}(d,k)} \\ =\ &Z_\chi(\sigma_k, 2n-k-s)\Gamma(n-k-s+1)^{-d_c(\chi)\, d(\sigma_k)}(2(n-k)-s)^{-d_c(\chi)\, \mathrm{d}(d,k)} \\ &\cdot \det C^k_\chi(\sigma_k, n-k-s)^{d(\sigma_k)}\ \det C^k_\chi(\sigma_k, 0)^{-d(\sigma_k)}\ \exp\Big(-\int_0^{s+k-n} \widetilde{P}_k(iz)\, dz\Big), \end{aligned}$$

$$\begin{aligned} &Z_\chi(\sigma_n, s+n)\Gamma(s+1)^{-2d_c(\chi)\, d(\sigma_n)} \\ =\ &Z_\chi(\sigma_n, n-s)\cdot\Gamma(-s+1)^{-2d_c(\chi)\, d(\sigma_n)} \\ &\cdot \det C^n_\chi(\sigma_n, -s)^{d(\sigma_n)} \det C^n_\chi(\sigma_n, 0)^{-d(\sigma_n)} \exp\Big(-\int_0^s 2\widetilde{P}_n(iz)\ dz\Big). \end{aligned}$$

By Proposition 7.4 and recalling

$$S_\chi(\ell) := \lim_{s\to -(n-\ell)} (s+n-\ell)^{-b_\ell} \det C^\ell_\chi(\sigma_\ell, s) = (-1)^{b_\ell} \lim_{s\to(n-\ell)} \Big((s-n+\ell)^{b_\ell} \det C^\ell_\chi(\sigma_\ell, s)\Big)^{-1},$$

we obtain

$$\begin{aligned} &Z_{2n-\ell}\ \exp\Big(\int_0^{n-\ell} \widetilde{P}_\ell(i\lambda)\, d\lambda\Big) \\ =&Z_\ell\left(\det C^\ell_\chi(\sigma_\ell,0) S_\chi(\ell)\right)^{d(\sigma_\ell)} \left(2(n-\ell)\right)^{d_c(\chi)\mathrm{d}(d,\ell)} \\ &\left((n-\ell)!\right)^{d_c(\chi) d(\sigma_\ell)} \left((n-\ell-1)!\right)^{d_c(\chi)d(\sigma_\ell)} (-1)^{\alpha_\ell + (n-\ell-1)d_c(\chi)d(\sigma_\ell)} \end{aligned} \tag{7.14}$$

where we also used the fact $\mathrm{res}_{z=-n}\Gamma(z) = \frac{(-1)^n}{n!}$. Combining (7.13) and (7.14),

$$\begin{aligned} \left(\det{}_\zeta \Delta_k\right)^2 = \prod_{\ell=k,k-1} &Z_{2n-\ell}\, Z_\ell \left(\det C^\ell_\chi(\sigma_\ell,0) S_\chi(\ell)\right)^{d(\sigma_\ell)} \left(2(n-\ell)\right)^{-2\alpha_\ell + d_c(\chi)\mathrm{d}(d,\ell)} \\ &(n-\ell)^{-d_c(\chi)(d(\sigma_\ell)+(-1)^{k-\ell}2\mathrm{d}(d,k))} e^{-2d_c(\chi)d(\sigma_\ell)C} (-1)^{\alpha_\ell+(n-\ell-1)d_c(\chi)d(\sigma_\ell)}. \end{aligned} \tag{7.15}$$

Using $\det_\zeta \Delta_k = \det_\zeta \Delta_{2n+1-k}$, we have

$$T(X_\Gamma, \chi) = (\det{}_\zeta\Delta_0)^{2n+1}\cdot(\det{}_\zeta\Delta_1)^{-(2n-1)}\cdot(\det{}_\zeta\Delta_2)^{(2n-3)}\dots(\det{}_\zeta\Delta_n)^{(-1)^n}. \tag{7.16}$$

From (6.14), we also have

$$\lim_{s\to 0}\left(s^{N_0} R_\chi(s)\right)^{-1} = \frac{Z_0}{Z_1}\frac{Z_2}{Z_3}\ \dots\ Z_{n-1}^{(-1)^{n-1}}\ Z_n^{(-1)^n}\ Z_{n+1}^{(-1)^{n+1}}\ \dots\ \frac{Z_{2n-2}}{Z_{2n-1}}\, Z_{2n}.$$

Now combining this and (7.15), (7.16) and recalling $\det C^\ell_\chi(\sigma_\ell, 0) = \pm 1$, finally we conclude that the following equality holds up to sign,

$$\lim_{s\to 0}\left(s^{N_0}R_\chi(s)\right)^{-1} = C(X_\Gamma,\chi)\cdot C(d)^{d_c(\chi)}\cdot S(X_\Gamma,\chi)\cdot T(X_\Gamma,\chi) \tag{7.17}$$

Here

$$C(X_\Gamma,\chi) := \prod_{k=0}^{n-1} \big(-4(n-k)^2\big)^{(-1)^k \alpha_k}, \qquad S(X_\Gamma,\chi) := \prod_{k=0}^{n-1} S_\chi(k)^{(-1)^{k+1} d(\sigma_k)}$$

and

$$C(d) := \prod_{k=0}^{n-1} 2^{(-1)^{k+1}\mathrm{d}(d,k)} \cdot (n-k)^{(-1)^k (2d(\sigma_k)(n-k)+\mathrm{d}(d,k))}.$$

Note that the terms $e^{-2d_c(\chi)d(\sigma_\ell)C}$'s are combined to be 1 by the equality

$$\sum_{\ell=0}^{n} (-1)^\ell d(\sigma_\ell) = 0.$$

8. Proof of Theorem 1.4

Throughout this section, we assume $d = 2n$.

As in the odd-dimensional case, we start with

$$M\big(\mathrm{Tr}_r(e^{-t\Delta_k}) - \beta_k\big) = MI_\chi(h_t^k) + MH_\chi(h_t^k) + MU_\chi(h_t^k) + MW_\chi(h_t^k) - M(\beta_k).$$

The Mellin transform of each term except $I_\chi(h_t^k)$ of this equality can be treated as in the odd-dimensional case. For $I_\chi(h_t^k)$, we have

Lemma 8.1. *When $d = 2n$, the following equality holds,*

$$\begin{aligned} MI_\chi(h_t^k)(s) =& \frac{\dim V_\chi \cdot \mathrm{vol}(\Gamma\backslash G)}{2^{2(2n-1)}\Gamma(n)^2} \\ & \cdot \sum_{\ell=k,k-1} d(\sigma_\ell) \sum_{j=0}^{n-1} b_j \frac{\Gamma(s)}{s-j-1} \int_0^\infty (x+d_\ell^2)^{-(s-j-1)} \frac{\pi \cosh^{-2}(\pi\sqrt{x})}{2\sqrt{x}}\, dx \end{aligned}$$

where the b_j's are given by $p(\sigma_\ell,\lambda) = \pi 2^{-4(n-1)}\Gamma(n)^{-2} d(\sigma_k)\lambda\tanh(\pi\lambda)\sum_{j=0}^{n-1} b_j(\lambda^2+d_\ell^2)^j$.

Proof. As in the derivation of (6.6), for $\mathrm{Re}(s) \gg 0$ we obtain

$$\begin{aligned} &\int_0^\infty t^{s-1} \int_{-\infty}^\infty e^{-t(\lambda^2+d_\ell^2)} \lambda\tanh(\pi\lambda)(\lambda^2+d_\ell^2)^j \, d\lambda\, dt \\ =&(s-1)\cdot \ldots \cdot (s-j) \int_0^\infty t^{s-j-2} \int_0^\infty e^{-t(x+d_\ell^2)} \Big(\frac{\pi\cosh^{-2}(\pi\sqrt{x})}{2\sqrt{x}}\Big)\, dx\, dt \\ =&\frac{\Gamma(s)}{s-j-1} \int_0^\infty (x+d_\ell^2)^{-(s-j-1)} \Big(\frac{\pi\cosh^{-2}(\pi\sqrt{x})}{2\sqrt{x}}\Big)\, dx, \end{aligned}$$

which, by analytic continuation, also holds for $s \in \mathbb{C}$. Since the term $\tilde{c}(\Gamma,\chi,\tau_n)$ in (6.7) is a constant with respect to t, its Mellin transform vanishes as in (7.2). Then this completes the proof. □

By Lemma 8.1, as also expected from Theorem 6.1, the limit of $MI_\chi(h_t^k)$ as $s \to 0$ does not exist by itself and we need to remove the simple pole of $MI_\chi(h_t^k)$ at $s = 0$. Lemma 8.1 immediately implies

Proposition 8.2.

$$\lim_{s\to 0}\big(MI_\chi(h_t^k)(s)-\Gamma(s)a_k\big) = \mathrm{dim}V_\chi\cdot \mathrm{vol}(\Gamma\backslash G)a(n,k)$$

where a_k is the residue of $MI_\chi(h_t^k)(s)$ at $s=0$ and $a(n,k)$ is a constant that is independent of Γ, but depends only on G.

Following [7], we write

$$a\sim b \qquad \text{if} \quad a=\exp(c\cdot \mathrm{dim}V_\chi\cdot\mathrm{vol}(\Gamma\backslash G))b \tag{8.1}$$

for a constant c that is independent of Γ. We can proceed as in the odd-dimensional case and obtain

$$\begin{aligned}\det{}_\zeta\Delta_k \sim \prod_{\ell=k,k-1} & Z_{2d_0-\ell}\,(2(d_0-\ell))^{-\alpha_\ell}\exp\big(\frac{1}{2}\int_0^{d_0-\ell}\widetilde{P}_\ell(i\lambda)\,d\lambda\big)\\ &\cdot (d_0-\ell)^{(-1)^{k-\ell+1}d_c(\chi)\mathrm{d}(d,k)}\Gamma(d_0-\ell+1)^{-d_c(\chi)d(\sigma_\ell)}e^{-d_c(\chi)d(\sigma_\ell)C}\end{aligned} \tag{8.2}$$

where $Z_{2d_0-\ell}$ denotes the leading coefficient of the Laurent expansion of $Z_\chi(\sigma_\ell,2d_0-\ell+x)$ at $x=0$ as in (7.8), $d_0=\frac{d-1}{2}$, $\widetilde{P}_\ell(s)=-d_c(\chi)P^d_\ell(s)+C(\chi,k)$ with the constant $C(\chi,k)$ from $U_\chi(h_t^k)$, which is determined by (4.14), (4.15). Now by Theorem 4.14 of [10],

Proposition 8.3. *For $s\in\mathbb{C}$, the following equality holds*

$$\begin{aligned}&Z_\chi(\sigma_k,s+k)\Gamma(s-d_0+k+1)^{-d_c(\chi)\,d(\sigma_k)}s^{-d_c(\chi)\,\mathrm{d}(d,k)}\Gamma_d(\sigma_k,s+k)\\ &= Z_\chi(\sigma_k,2d_0-k-s)\Gamma(d_0-k-s+1)^{-d_c(\chi)\,d(\sigma_k)}(2(d_0-k)-s)^{-d_c(\chi)\,\mathrm{d}(d,k)}\Gamma_d(\sigma_k,2d_0-k-s)\\ &\quad\cdot \det C^k_\chi(\sigma_k,d_0-k-s)^{d(\sigma_k)}\ \det C^k_\chi(\sigma_k,0)^{-d(\sigma_k)}\ \exp\Big(-\int_0^{s+k-d_0}\widetilde{P}_k(iz)\,dz\Big)\end{aligned}$$

where

$$\Gamma_d(\sigma_k,s)=\Big[\prod_{\ell=0}^{k}\Big(\Gamma_d(s-\ell)\Gamma_d(s+\ell+1)\Big)^{(-1)^\ell\binom{d}{k-\ell}}\Big]^{-\dim V_\chi\,E(X_\Gamma)}.$$

Here $\Gamma_d(s)$ is the multiple gamma function of order d introduced in [15] *and $E(X_\Gamma)$ denotes the Euler characteristic of X_Γ.*

From Proposition 8.3,

$$\begin{aligned}&Z_{2d_0-\ell}\exp\big(\int_0^{d_0-\ell}\widetilde{P}_\ell(i\lambda)\,d\lambda\big)\\ &\sim Z_\ell\,\big(\det C^\ell_\chi(\sigma_\ell,0)S_\chi(\ell)\big)^{d(\sigma_\ell)}\,\big(2(d_0-\ell)\big)^{d_c(\chi)\mathrm{d}(d,\ell)}\\ &\quad\Gamma(d_0-\ell+1)^{d_c(\chi)d(\sigma_\ell)}\,\Gamma(-d_0+\ell+1)^{-d_c(\chi)d(\sigma_\ell)}\,(-1)^{\alpha_\ell}\end{aligned} \tag{8.3}$$

where

$$S_\chi(\ell):=\lim_{s\to-(d_0-\ell)}(s+d_0-\ell)^{-b_\ell}\det C^\ell_\chi(\sigma_\ell,s).$$

The ambiguity '$\sim$' in (8.3) comes from the constant term of the Laurent expansion of $\Gamma_d(\sigma_\ell,2d_0-s)\Gamma_d(\sigma_\ell,s)^{-1}$ at $s=\ell$ and the following equality given in Proposition 4.4 of [9],

$$\mathrm{vol}(\Gamma\backslash G)\frac{n}{2^{4n-3}}\binom{2n-1}{n}=(-1)^n\,E(X_\Gamma).$$

Combining (8.2) and (8.3),

$$(8.4)\quad \big(\det{}_\zeta\Delta_k\big)^2 \sim \prod_{\ell=k,k-1} Z_{2d_0-\ell}\, Z_\ell\, \big(\det C^\ell_\chi(\sigma_\ell,0)S_\chi(\ell)\big)^{d(\sigma_\ell)}$$
$$(-1)^{\alpha_\ell}\big(2(d_0-\ell)\big)^{-2\alpha_\ell+d_c(\chi)\mathrm{d}(d,\ell)}\,\big(\Gamma(d_0-\ell+1)\Gamma(-d_0+\ell+1)\big)^{-d_c(\chi)d(\sigma_\ell)}$$
$$(d_0-\ell)^{(-1)^{k-\ell+1}2d_c(\chi)\mathrm{d}(d,k)}e^{-2d_c(\chi)d(\sigma_\ell)C}.$$

Using $\det_\zeta\Delta_k = \det_\zeta\Delta_{2n-k}$, we have

$$(8.5)\quad T(X_\Gamma,\chi) = (\det{}_\zeta\Delta_0)^{-2n}\cdot(\det{}_\zeta\Delta_1)^{2n}\cdot(\det{}_\zeta\Delta_2)^{-2n}\ldots(\det{}_\zeta\Delta_{n-1})^{(-1)^n 2n}\cdot(\det{}_\zeta\Delta_n)^{(-1)^{n+1}n}.$$

By (8.4) and the symmetry in (8.5), all terms cancel except the term $(d_0-\ell)^{(-1)^{k-\ell+1}2d_c(\chi)\mathrm{d}(d,k)}$ if we plug (8.4) into (8.5). For instance, the leading terms Z_ℓ of $Z_\chi(\sigma_\ell,\ell)$ combined to

$$(8.6)\quad (Z_0Z_{2n-1})^{-n}\cdot(Z_1Z_{2n-2})^{n}\cdot(Z_0Z_{2n-1})^{n}\ldots(Z_{n-1}Z_n)^{(-1)^n n}\cdot(Z_{n-2}Z_{n+1})^{(-1)^n n}\cdot(Z_{n-1}Z_n)^{(-1)^{n+1}n}=1.$$

The combination of terms $(d_0-\ell)^{(-1)^{k-\ell+1}2d_c(\chi)\mathrm{d}(d,k)}$ results in

$$T(X_\Gamma,\chi)\sim\Big(\prod_{k=0}^{n-1}(d_0-k)^{(-1)^k n(\binom{2n-1}{k+1}-\binom{2n-2}{k})}\Big)^{d_c(\chi)}.$$

Recalling the definition in (8.1),

$$(8.7)\quad T(X_\Gamma,\chi)=\exp(a(G)\dim V_\chi\cdot\mathrm{vol}(\Gamma\backslash G))\Big(\prod_{k=0}^{n-1}(d_0-k)^{(-1)^k n(\binom{2n-1}{k+1}-\binom{2n-2}{k})}\Big)^{d_c(\chi)}$$

where $a(G)$ is a constant depending only on G, not on Γ. Now let us observe that the equality (8.7) still holds with $d_c(\chi)=0$ if X_Γ is compact, that is,

$$1=T(X_\Gamma,\chi)=\exp(a(G)\dim V_\chi\cdot\mathrm{vol}(\Gamma\backslash G))$$

for any co-compact discrete group $\Gamma\subset G$. Hence it follows that the constant $a(G)=0$. Finally we conclude

$$(8.8)\quad T(X_\Gamma,\chi)=\Big(\prod_{k=0}^{n-1}(d_0-k)^{(-1)^k n\big(\binom{2n-1}{k+1}-\binom{2n-2}{k}\big)}\Big)^{d_c(\chi)}.$$

Appendix A. Proof of Theorem 5.3

A.1. **Odd dimensional case:** $d=2n+1$**.** The case of $n=1$ can be computed as in the cases $n\geq 2$. Hence we assume that $n\geq 2$ in the following proof. The highest weights $\mu_k,\mu_n^\pm$ of the representations σ_k, $\sigma_n^\pm$ of $M=\mathrm{SO}(2n)\subset K=\mathrm{SO}(2n+1)$ are given by

$$\mu_k=e_2+e_3+\cdots+e_{k+1}\quad(0\leq k\leq n-1),\qquad \mu_n^\pm=e_2+e_3+\cdots+e_n\pm e_{n+1}.$$

Recalling

$$\rho_M=(n-1)e_2+(n-2)e_3+\ldots+e_n,$$

we have

$$\begin{aligned} &\lambda_{\sigma_k} = i\lambda e_1 + \mu_k + \rho_M \\ =& i\lambda e_1 + ne_2 + (n-1)e_3 + \cdots + (n-k+1)e_{k+1} + (n-k-1)e_{k+2} \cdots + e_n, \\ &\lambda_{\sigma_n^\pm} = i\lambda e_1 + \mu_n^\pm + \rho_M \\ =& i\lambda e_1 + ne_2 + (n-1)e_3 + \cdots + 2e_n \pm e_{n+1}. \end{aligned}$$

First we consider $\Pi(s_\alpha \lambda_\sigma)$ for $\alpha \in \Sigma_A$, which are given by $e_1 - e_\ell, e_1 + e_\ell$ for $2 \leq \ell \leq n+1$. Then we have

$$\begin{aligned} & s_{(e_1-e_\ell)}(i\lambda e_1 + \mu_k + \rho_M) \\ =& \begin{cases} i\lambda e_\ell + ne_2 + \cdots + (n-\ell+2)e_1 + \cdots + (n-k+1)e_{k+1} & \\ \qquad\qquad\qquad + (n-k-1)e_{k+2} + \cdots + e_n & \text{if} \quad 2 \leq \ell \leq k+1, \\ i\lambda e_\ell + ne_2 + \cdots + (n-k+1)e_{k+1} & \\ \qquad\qquad + (n-k-1)e_{k+2} + \cdots + (n-\ell+1)e_1 + \cdots + e_n & \text{if} \quad k+2 \leq \ell \leq n, \\ i\lambda e_{n+1} + ne_2 + \cdots + (n-k+1)e_{k+1} + (n-k-1)e_{k+2} + \cdots + e_n & \text{if} \quad \ell = n+1, \end{cases} \end{aligned}$$

$$\begin{aligned} & s_{(e_1-e_\ell)}(i\lambda e_1 + \mu_n^\pm + \rho_M) \\ =& \begin{cases} i\lambda e_\ell + ne_2 + \cdots + (n-\ell+2)e_1 + \cdots + 2e_n \pm e_{n+1} & \text{if} \quad 2 \leq \ell \leq n, \\ i\lambda e_{n+1} + ne_2 + \cdots + 2e_n \pm e_1 & \text{if} \quad \ell = n+1, \end{cases} \end{aligned}$$

$$\begin{aligned} & s_{(e_1+e_\ell)}(i\lambda e_1 + \mu_k + \rho_M) \\ =& \begin{cases} -i\lambda e_\ell + ne_2 + \cdots - (n-\ell+2)e_1 + \cdots + (n-k+1)e_{k+1} & \\ \qquad\qquad\qquad + (n-k-1)e_{k+2} + \cdots + e_n & \text{if} \quad 2 \leq \ell \leq k+1, \\ -i\lambda e_j + ne_2 + \cdots + (n-k+1)e_{k+1} & \\ \qquad\qquad + (n-k-1)e_{k+2} + \cdots - (n-\ell+1)e_1 + \cdots + e_n & \text{if} \quad k+2 \leq \ell \leq n, \\ -i\lambda e_{n+1} + ne_2 + \cdots + (n-k+1)e_{k+1} + (n-k-1)e_{k+2} + \cdots + e_n & \text{if} \quad \ell = n+1, \end{cases} \end{aligned}$$

$$\begin{aligned} & s_{(e_1+e_\ell)}(i\lambda e_1 + \mu_n^\pm + \rho_M) \\ =& \begin{cases} -i\lambda e_\ell + ne_2 + \cdots - (n-\ell+2)e_1 + \cdots + 2e_n \pm e_{n+1} & \text{if} \quad 2 \leq \ell \leq n, \\ -i\lambda e_{n+1} + ne_2 + \cdots + 2e_n \mp e_1 & \text{if} \quad \ell = n+1. \end{cases} \end{aligned}$$

The above computations give us the following equalities,

$$\begin{aligned} & \Pi(s_{(e_1 \pm e_\ell)}(i\lambda e_1 + \mu_k + \rho_M)) \\ =& C^k_{\ell-1}(\lambda^2 + n^2) \cdot (\lambda^2 + (n-1)^2) \cdots (\lambda^2 + (n-\ell+3)^2) \cdot (-\lambda^2 - (n-\ell+1)^2) \cdots \\ & (-\lambda^2 - (n-k+1)^2) \cdot (-\lambda^2 - (n-k-1)^2) \cdots (-\lambda^2) \qquad \text{if} \quad 2 \leq \ell \leq k+1, \end{aligned}$$

$$\begin{aligned} & \Pi(s_{(e_1 \pm e_\ell)}(i\lambda e_1 + \mu_k + \rho_M)) \\ =& C^k_{\ell}(\lambda^2 + n^2) \cdot (\lambda^2 + (n-1)^2) \cdots (\lambda^2 + (n-k+1)^2) \cdot (\lambda^2 + (n-k-1)^2) \cdots \\ & (\lambda^2 + (n-\ell+2)^2) \cdot (-\lambda^2 - (n-\ell)^2) \cdots (-\lambda^2) \qquad \text{if} \quad k+2 \leq \ell \leq n+1, \end{aligned}$$

$$\begin{aligned}&\Pi(s_{(e_1\pm e_\ell)}(i\lambda e_1+\mu_n^\pm+\rho_M))\\ &= C_{\ell-1}^n(\lambda^2+n^2)\cdot(\lambda^2+(n-1)^2)\cdots(\lambda^2+(n-\ell+3)^2)\cdot(-\lambda^2-(n-\ell+1)^2)\\ &\quad\cdots(-\lambda^2-2^2)(-\lambda^2-1)\end{aligned}$$

where

$$C_\ell^k = \prod_{\substack{0\le a<b\le n\\ a,b\neq(n-k),(n-\ell+1)}}(b^2-a^2)$$

for $0\le k\le n$, $2\le\ell\le n+1$. By the above computation, we can put

$$P_{k,\ell}^n := \Pi(s_{e_1\pm e_\ell}(i\lambda e_1+\mu_k+\rho_M)),$$
$$P_{n,\ell}^n := \Pi(s_{e_1\pm e_\ell}(i\lambda e_1+\mu_n^\pm+\rho_M)),$$

which are degree $2(n-1)$ even polynomials of λ.

Second, we compute the part $(\psi(1+\lambda_\sigma(H_\alpha))+\psi(1-\lambda_\sigma(H_\alpha))$. To do so, note that

$$(i\lambda e_1+\mu_k+\rho_M)(H_\alpha)=\begin{cases} i\lambda-(n-\ell+2) & \text{if} \quad \alpha=e_1-e_\ell,\quad 2\le\ell\le k+1\\ i\lambda-(n-\ell+1) & \text{if} \quad \alpha=e_1-e_\ell,\quad k+2\le\ell\le n\\ i\lambda & \text{if} \quad \alpha=e_1-e_{n+1}\\ i\lambda+(n-\ell+2) & \text{if} \quad \alpha=e_1+e_\ell,\quad 2\le\ell\le k+1\\ i\lambda+(n-\ell+1) & \text{if} \quad \alpha=e_1+e_\ell,\quad k+2\le\ell\le n\\ i\lambda & \text{if} \quad \alpha=e_1+e_{n+1},\end{cases}$$

$$(i\lambda e_1+\mu_n^\pm+\rho_M)(H_\alpha)=\begin{cases} i\lambda-(n-\ell+2) & \text{if} \quad \alpha=e_1-e_\ell,\quad 2\le\ell\le n\\ i\lambda\mp1 & \text{if} \quad \alpha=e_1-e_{n+1}\\ i\lambda+(n-\ell+2) & \text{if} \quad \alpha=e_1+e_\ell,\quad 2\le\ell\le n\\ i\lambda\pm1 & \text{if} \quad \alpha=e_1+e_{n+1}.\end{cases}$$

From these equalities, we can see that $(\psi(1+(i\lambda e_1+\mu_k+\rho_M)(H_\alpha)),\psi(1-(i\lambda e_1+\mu_k+\rho_M)(H_\alpha))$ is given by

$$\begin{array}{lll} \psi(i\lambda-n+\ell-1),\ \psi(-i\lambda+n-\ell+3) & \text{for} & \alpha=e_1-e_\ell,\quad 2\le\ell\le k+1\\ \psi(i\lambda-n+\ell),\ \psi(-i\lambda+n-\ell+2) & \text{for} & \alpha=e_1-e_\ell,\quad k+2\le\ell\le n\\ \psi(i\lambda+1),\ \psi(-i\lambda+1) & \text{for} & \alpha=e_1-e_{n+1},\\ \psi(i\lambda+n-\ell+3),\ \psi(-i\lambda-n+\ell-1) & \text{for} & \alpha=e_1+e_\ell,\quad 2\le\ell\le k+1\\ \psi(i\lambda+n-\ell+2),\ \psi(-i\lambda-n+\ell) & \text{for} & \alpha=e_1+e_\ell,\quad k+2\le\ell\le n\\ \psi(i\lambda+1),\ \psi(-i\lambda+1) & \text{for} & \alpha=e_1+e_{n+1},\end{array}$$

and $(\psi(1+(i\lambda e_1+\mu_n^\pm+\rho_M)(H_\alpha)),\psi(1-(i\lambda e_1+\mu_n^\pm+\rho_M)(H_\alpha))$ is given by

$$\begin{array}{lll} \psi(i\lambda-n+\ell-1),\ \psi(-i\lambda+n-\ell+3) & \text{for} & \alpha=e_1-e_\ell,\quad 2\le\ell\le n\\ \psi(i\lambda),\ \psi(-i\lambda+2) & \text{for} & \alpha=e_1-e_{n+1},\quad \sigma=\sigma_+\\ \psi(i\lambda+2),\ \psi(-i\lambda) & \text{for} & \alpha=e_1-e_{n+1},\quad \sigma=\sigma_-\\ \psi(i\lambda+n-\ell+3),\ \psi(-i\lambda-n+\ell-1) & \text{for} & \alpha=e_1+e_\ell,\quad 2\le\ell\le n\\ \psi(i\lambda+2),\ \psi(-i\lambda) & \text{for} & \alpha=e_1+e_{n+1},\quad \sigma=\sigma_+\\ \psi(i\lambda),\ \psi(-i\lambda+2) & \text{for} & \alpha=e_1+e_{n+1},\quad \sigma=\sigma_-.\end{array}$$

Putting

$$\Psi_n(i\lambda) := \psi(i\lambda - n) + \psi(-i\lambda - n) + \psi(i\lambda + 1) + \psi(-i\lambda + 1),$$

we have

$$\begin{aligned}
&\psi(i\lambda - n + \ell - 1) + \psi(-i\lambda - n + \ell - 1) + \psi(i\lambda + n - \ell + 3) + \psi(-i\lambda + n - \ell + 3) \\
= {}& \Psi_n(i\lambda) + \frac{2}{\lambda^2+1} + \cdots + \frac{2(n-\ell+1)}{\lambda^2 + (n-\ell+1)^2} + \frac{-2(n-\ell+3)}{\lambda^2+(n-\ell+3)^2} + \cdots + \frac{-2n}{\lambda^2+n^2}, \\
&\psi(i\lambda - n + \ell) + \psi(-i\lambda - n + \ell) + \psi(i\lambda + n - \ell + 2) + \psi(-i\lambda + n - \ell + 2) \\
= {}& \Psi_n(i\lambda) + \frac{2}{\lambda^2+1} + \cdots + \frac{2(n-\ell)}{\lambda^2 + (n-\ell)^2} + \frac{-2(n-\ell+2)}{\lambda^2+(n-\ell+2)^2} + \cdots + \frac{-2n}{\lambda^2+n^2}, \\
&2(\psi(i\lambda+1) + \psi(-i\lambda+1)) = \Psi_n(i\lambda) + \frac{-2}{\lambda^2+1} + \cdots + \frac{-2n}{\lambda^2+n^2}, \\
&\psi(i\lambda+2) + \psi(-i\lambda+2) + \psi(i\lambda) + \psi(-i\lambda) = \Psi_n(i\lambda) + \frac{-2\cdot 2}{\lambda^2+2^2} + \cdots + \frac{-2n}{\lambda^2+n^2}.
\end{aligned}$$

Now we assume that $0 \le k \le n-1$. Using the formula $\sum_{\alpha\in\Sigma_A} \Pi(s_\alpha(\lambda_{\sigma_k})) = 2\Pi(\lambda_{\sigma_k})$ (see the last line on p. 95 in [13]) and the above formulas to decompose

$$\frac{1}{2}\sum_{\alpha\in\Sigma_A} \frac{\Pi(s_\alpha\lambda_{\sigma_k})}{\Pi(\rho_M)} \times \Big(\psi(1+\lambda_{\sigma_k}(H_\alpha)) + \psi(1-\lambda_{\sigma_k}(H_\alpha))\Big)$$

into

$$\frac{d(\sigma_k)}{2}\Psi_n(i\lambda) = \frac{d(\sigma_k)}{2}\Big(\ \psi(i\lambda - n) + \psi(-i\lambda - n) + \psi(i\lambda+1) + \psi(-i\lambda+1)\ \Big)$$

(where we use the Weyl's dimension formula for $d(\sigma_k)$) and

$$\frac{1}{2\Pi(\rho_M)}\Big(\sum_{\ell=2}^{k+1} P^n_{k,\ell}(\lambda) Q_{n,\ell}(\lambda) + \sum_{\ell=k+2}^{n+1} P^n_{k,\ell}(\lambda) R_{n,\ell}(\lambda)\Big)$$

where

$$\begin{aligned}
Q_{n,\ell}(\lambda) &= \frac{2}{\lambda^2+1} + \cdots + \frac{2(n-\ell+1)}{\lambda^2+(n-\ell+1)^2} + \frac{-2(n-\ell+3)}{\lambda^2+(n-\ell+3)^2} + \cdots + \frac{-2n}{\lambda^2+n^2}, \\
R_{n,\ell}(\lambda) &= \frac{2}{\lambda^2+1} + \cdots + \frac{2(n-\ell)}{\lambda^2+(n-\ell)^2} + \frac{-2(n-\ell+2)}{\lambda^2+(n-\ell+2)^2} + \cdots + \frac{-2n}{\lambda^2+n^2}.
\end{aligned}$$

By the definitions of $P^n_{k,\ell}(\lambda)$, $Q_{n,\ell}(\lambda)$ and $R_{n,\ell}(\lambda)$, we can see that $P^n_{k,\ell}(\lambda)Q_{n,\ell}(\lambda)$, (or $P^n_{k,\ell}(\lambda)R_{n,\ell}(\lambda)$) is the sum of even polynomials of degree $2n-4$, which is the polynomial $P^d_k(\lambda)$ in Theorem 5.3, and

$$\mathcal{R}^n_k := \frac{1}{\Pi(\rho_M)}\frac{(n-k)}{\lambda^2+(n-k)^2}\Big(\sum_{2\le\ell\le k+1} q^n_{k,\ell} + \sum_{k+2\le\ell\le n+1} r^n_{k,\ell}\Big)$$

where

$$\begin{aligned}
q^n_{k,\ell} &= (-1)^{k+\ell+1} \prod_{\substack{0\le a<b\le n \\ a,b\ne n-\ell+2}} (b^2-a^2), \\
r^n_{k,\ell} &= (-1)^{k+\ell+1} \prod_{\substack{0\le a<b\le n \\ a,b\ne n-\ell+1}} (b^2-a^2).
\end{aligned}$$

By the Weyl's multiplicity formula for $d(\sigma_k)$, we have

$$\mathcal{R}^n_k \;=\; \frac{(-1)^{k+1}(n-k)}{\lambda^2+(n-k)^2}\;\Big(\sum_{j=0}^{k-1}(-1)^j d(\sigma_j)+\sum_{j=k+1}^{2n}(-1)^{j+1}d(\sigma_j)\Big)$$

where we also use

$$\sum_{j=0}^{n}(-1)^j d(\sigma_j)=\sum_{j=n}^{2n}(-1)^{j+1}d(\sigma_j).$$

By an essentially same computation, we decompose

$$\frac{1}{2}\sum_{\alpha\in\Sigma_A}\frac{\Pi(s_\alpha\lambda_{\sigma_\pm})}{\Pi(\rho_M)}\times\Big(\psi(1+\lambda_{\sigma_\pm}(H_\alpha))+\psi(1-\lambda_{\sigma_\pm}(H_\alpha))\Big)$$

into

$$\frac{d(\sigma_k)}{2}\Psi_n(i\lambda)=\frac{d(\sigma_\pm)}{2}\Big(\;\psi(i\lambda-n)+\psi(-i\lambda-n)+\psi(i\lambda+1)+\psi(-i\lambda+1)\;\Big)$$

and

$$\frac{1}{2\Pi(\rho_M)}\sum_{\ell=2}^{n+1}P^n_{n,\ell}(\lambda)Q_{n,\ell}(\lambda).$$

By the definitions of $P^n_{n,\ell}(\lambda)$ and $Q_{n,\ell}(\lambda)$, we can see that $P^n_{n,\ell}(\lambda)Q_{n,\ell}(\lambda)$ is an even polynomial of degree $2n-4$, which we can denote by $P^d_n(\lambda)$.

A.2. **Even dimensional case:** $d=2(n+1)$**.** For convenience of the computation, we let $n=\frac{d}{2}-1$ so that $d=2(n+1)$ throughout this subsection. The case of $n=0$ can be computed as in the cases $n\geq 1$. Hence we assume that $n\geq 1$ in the following proof.

With respect to the inner product on $\mathfrak{t}^*_{\mathbb{C}}$ induced from $\langle\cdot,\cdot\rangle$ in (2.1), we choose an orthonormal basis $\{e_i\}$ of $\mathfrak{t}^*_{\mathbb{C}}$ such that $e_1\in\mathfrak{a}^*_{\mathbb{C}}$. Then we have

$$\Sigma_G=\{\ e_i\ (1\le i\le n+1),\quad e_i-e_j\ (1\le i<j\le n+1),\quad e_i+e_j\ (1\le i<j\le n+1)\ \},$$
$$\Sigma_A=\{\ e_1,\quad e_1-e_j\ (1<j\le n+1),\quad e_1+e_j\ (1<j\le n+1)\ \}.$$

Let us write λ_{σ_k} in terms of $\{e_i\}$. The highest weights μ_k of the representations σ_k of $M=\mathrm{SO}(2n+1)\subset K=\mathrm{SO}(2(n+1))$ are given by

$$\mu_k=e_2+e_3+\cdots+e_{k+1}\quad(0\le k\le n).$$

Recalling

$$\rho_M=(n-\frac{1}{2})e_2+(n-\frac{3}{2})e_3+\ldots+\frac{1}{2}e_{n+1},$$

we have

$$\begin{aligned}\lambda_{\sigma_k}=&i\lambda e_1+\mu_k+\rho_M\\ =&i\lambda e_1+(n+\frac{1}{2})e_2+(n-\frac{1}{2})e_3+\cdots+(n-k+\frac{3}{2})e_{k+1}+(n-k-\frac{1}{2})e_{k+2}\cdots+\frac{1}{2}e_{n+1}.\end{aligned}$$

First we consider $\Pi(s_\alpha\lambda_\sigma)$ for $\alpha\in\Sigma_A$, which are given by $e_1, e_1-e_\ell, e_1+e_\ell$ for $2\le\ell\le n+1$. Then we have

$$\begin{aligned}&s_{e_1}(i\lambda e_1+\mu_k+\rho_M)\\ =&-i\lambda e_1+(n+\frac{1}{2})e_2+(n-\frac{1}{2})e_3+\cdots+(n-k+\frac{3}{2})e_{k+1}+(n-k-\frac{1}{2})e_{k+2}\cdots+\frac{1}{2}e_{n+1}.\end{aligned}$$

$$s_{(e_1-e_\ell)}(i\lambda e_1+\mu_k+\rho_M)$$
$$=\begin{cases} i\lambda e_\ell+(n+\frac12)e_2+\cdots+(n-\ell+\frac52)e_1+\cdots+(n-k+\frac32)e_{k+1} \\ \qquad\qquad +(n-k-\frac12)e_{k+2}+\cdots+\frac12 e_{n+1} & \text{if} \quad 2\le \ell\le k+1, \\ i\lambda e_\ell+(n+\frac12)e_2+\cdots+(n-k+\frac32)e_{k+1} \\ \qquad +(n-k-\frac12)e_{k+2}+\cdots+(n-\ell+\frac32)e_1+\cdots+\frac12 e_{n+1} & \text{if} \quad k+2\le \ell\le n+1, \end{cases}$$

$$s_{(e_1+e_\ell)}(i\lambda e_1+\mu_k+\rho_M)$$
$$=\begin{cases} -i\lambda e_\ell+(n+\frac12)e_2+\cdots-(n-\ell+\frac52)e_1+\cdots+(n-k+\frac32)e_{k+1} \\ \qquad\qquad +(n-k-\frac12)e_{k+2}+\cdots+\frac12 e_{n+1} & \text{if} \quad 2\le \ell\le k+1, \\ -i\lambda e_\ell+(n+\frac12)e_2+\cdots+(n-k+\frac32)e_{k+1} \\ \qquad +(n-k-\frac12)e_{k+2}+\cdots-(n-\ell+\frac32)e_1+\cdots+\frac12 e_{n+1} & \text{if} \quad k+2\le \ell\le n+1. \end{cases}$$

Recall that Σ_M consists of e_i for $2\le i\le n+1$, $e_i\pm e_j$ for $2\le i<j\le n+1$ and the co-root H_α of α satisfies $\alpha(H_\alpha)=2$. By the Weyl's dimension formula, for $\alpha=e_1$,

$$\Pi\big(s_{e_1}(i\lambda e_1+\mu_k+\rho_M)\big)=d(\sigma_k)\Pi(\rho_M).$$

For the other cases, it is a polynomial of λ as follows:

$$\begin{aligned} &\Pi(s_{(e_1\pm e_\ell)}(i\lambda e_1+\mu_k+\rho_M)) \\ &= C^k_{\ell-1}(\mp i\lambda)(\lambda^2+(n+\frac12)^2)\cdots(\lambda^2+(n-\ell+\frac72)^2)\cdot(-\lambda^2-(n-\ell+\frac32)^2)\cdots \\ &\quad (-\lambda^2-(n-k+\frac32)^2)\cdot(-\lambda^2-(n-k-\frac12)^2)\cdots(-\lambda^2-(\frac12)^2) \qquad \text{if} \quad 2\le\ell\le k+1, \end{aligned}$$

$$\begin{aligned} &\Pi(s_{(e_1\pm e_\ell)}(i\lambda e_1+\mu_k+\rho_M)) \\ &= C^k_{\ell}(\mp i\lambda)(\lambda^2+(n+\frac12)^2)\cdots(\lambda^2+(n-k+\frac32)^2)\cdot(\lambda^2+(n-k-\frac12)^2)\cdots \\ &\quad (\lambda^2+(n-\ell+\frac52)^2)\cdot(-\lambda^2-(n-\ell+\frac12)^2)\cdots(-\lambda^2-(\frac12)^2) \qquad \text{if} \quad k+2\le\ell\le n+1 \end{aligned}$$

where

$$C^k_\ell=2^n\prod_{\substack{0\le a<b\le n\\ a,b\notin\{n-k,n-\ell\}}}\big((b+\frac12)^2-(a+\frac12)^2\big)\cdot\prod_{\substack{0\le c\le n\\ c\notin\{n-k,n-\ell\}}}(c+\frac12)$$

for $0\le k\le n$, $2\le\ell\le n+1$. By the above computation, we can put

$$P^n_{k,\ell}(\lambda):=\Pi(s_{e_1\pm e_\ell}(i\lambda e_1+\mu_k+\rho_M))$$

which is degree $2n-3$ odd polynomial of λ.

Second we compute the part $(\psi(1+\lambda_\sigma(H_\alpha))+\psi(1-\lambda_\sigma(H_\alpha))$ for $\alpha\in\Sigma_A$. For this,

$$(i\lambda e_1+\mu_k+\rho_M)(H_\alpha)=\begin{cases} 2i\lambda & \text{if} \quad \alpha=2e_1 \\ i\lambda-(n-\ell+\frac52) & \text{if} \quad \alpha=e_1-e_\ell, \quad 2\le\ell\le k+1 \\ i\lambda-(n-\ell+\frac32) & \text{if} \quad \alpha=e_1-e_\ell, \quad k+2\le\ell\le n+1 \\ i\lambda+(n-\ell+\frac52) & \text{if} \quad \alpha=e_1+e_\ell, \quad 2\le\ell\le k+1 \\ i\lambda+(n-\ell+\frac32) & \text{if} \quad \alpha=e_1+e_\ell, \quad k+2\le\ell\le n+1. \end{cases}$$

From this, we can see that $(\psi(1+(i\lambda e_1+\mu_k+\rho_M)(H_\alpha)),\psi(1-(i\lambda e_1+\mu_k+\rho_M)(H_\alpha))$ is given by

$$\begin{aligned}
&\psi(2i\lambda+1),\ \psi(-2i\lambda+1) && \text{for}\quad \alpha=2e_1,\\
&\psi(i\lambda-n+\ell-\frac{3}{2}),\ \psi(-i\lambda+n-\ell+\frac{7}{2}) && \text{for}\quad \alpha=e_1-e_\ell,\quad 2\le\ell\le k+1\\
&\psi(i\lambda-n+\ell-\frac{1}{2}),\ \psi(-i\lambda+n-\ell+\frac{5}{2}) && \text{for}\quad \alpha=e_1-e_\ell,\quad k+2\le\ell\le n+1\\
&\psi(i\lambda+n-\ell+\frac{7}{2}),\ \psi(-i\lambda-n+\ell-\frac{3}{2}) && \text{for}\quad \alpha=e_1+e_\ell,\quad 2\le\ell\le k+1\\
&\psi(i\lambda+n-\ell+\frac{5}{2}),\ \psi(-i\lambda-n+\ell-\frac{1}{2}) && \text{for}\quad \alpha=e_1+e_\ell,\quad k+2\le\ell\le n+1.
\end{aligned}$$

For the sum over $\alpha\in\Sigma_A$ in (5.5), we first consider the term with $\alpha=e_1$. By the results obtained above,

$$\begin{aligned}
&\frac{1}{2}\beta(H_{e_1})\frac{\Pi(s_{e_1}\lambda_\sigma)}{\Pi(\rho_M)}\Big(\psi(1+\lambda_\sigma(H_{e_1}))+\psi(1-\lambda_\sigma(H_{e_1}))\Big)\\
&=d(\sigma_k)\Big(\psi(2i\lambda+1)+\psi(-2i\lambda+1)\Big)\\
&=\frac{d(\sigma_k)}{2}\Big(\psi(i\lambda+\frac{1}{2})+\psi(-i\lambda+\frac{1}{2})+\psi(i\lambda+1)+\psi(-i\lambda+1)+4\log 2\Big)\\
&=\frac{d(\sigma_k)}{2}\Big(\psi(i\lambda-n-\frac{1}{2})+\psi(-i\lambda-n-\frac{1}{2})+\psi(i\lambda+1)+\psi(-i\lambda+1)\\
&\qquad+\frac{-2\cdot\frac{1}{2}}{\lambda^2+(\frac{1}{2})^2}+\ldots+\frac{-2(n+\frac{1}{2})}{\lambda^2+(n+\frac{1}{2})^2}+4\log 2\Big)
\end{aligned}\tag{A.1}$$

by the properties of the digamma function $\psi(z)$. Now we take a sum over e_1+e_ℓ, e_1-e_ℓ in (5.5). For $2\le\ell\le k+1$,

$$\begin{aligned}
&\frac{1}{2}\sum_{\alpha=e_1\pm e_\ell}\beta(H_\alpha)\frac{\Pi(s_\alpha\lambda_\sigma)}{\Pi(\rho_M)}\Big(\psi(1+\lambda_\sigma(H_\alpha))+\psi(1-\lambda_\sigma(H_\alpha))\Big)\\
&=\frac{1}{2}\frac{\Pi(s_{e_1-e_\ell}\lambda_\sigma)}{\Pi(\rho_M)}\Big(\psi(i\lambda-n+\ell-\frac{3}{2})+\psi(-i\lambda+n-\ell+\frac{7}{2})\\
&\qquad-\psi(i\lambda+n-\ell+\frac{7}{2})-\psi(-i\lambda-n+\ell-\frac{3}{2})\Big)\\
&=\frac{1}{2}\frac{\Pi(s_{e_1-e_\ell}\lambda_\sigma)}{\Pi(\rho_M)}\Big(\frac{4i\lambda}{\lambda^2+(\frac{1}{2})^2}+\ldots+\frac{4i\lambda}{\lambda^2+(n-\ell+\frac{3}{2})^2}+\frac{2i\lambda}{\lambda^2+(n-\ell+\frac{5}{2})^2}\Big),
\end{aligned}\tag{A.2}$$

and similarly for $k+2\le\ell\le n+1$,

$$\begin{aligned}
&\frac{1}{2}\sum_{\alpha=e_1\pm e_\ell}\beta(H_\alpha)\frac{\Pi(s_\alpha\lambda_\sigma)}{\Pi(\rho_M)}\Big(\psi(1+\lambda_\sigma(H_\alpha))+\psi(1-\lambda_\sigma(H_\alpha))\Big)\\
&=\frac{1}{2}\frac{\Pi(s_{e_1-e_\ell}\lambda_\sigma)}{\Pi(\rho_M)}\Big(\frac{4i\lambda}{\lambda^2+(\frac{1}{2})^2}+\ldots+\frac{4i\lambda}{\lambda^2+(n-\ell+\frac{1}{2})^2}+\frac{2i\lambda}{\lambda^2+(n-\ell+\frac{3}{2})^2}\Big).
\end{aligned}\tag{A.3}$$

From the expression of $\Pi(s_{e_1-e_\ell}\lambda_\sigma)$, we can see that the term in (A.2), (A.3) consists of a polynomial of degree $2n-2$ if $d=2(n+1)\ge 4$ and some rational functions whose denominators are $\lambda^2+(n-k+\frac{1}{2})^2$, $\lambda^2+(n-\ell+\frac{5}{2})^2$ when $2\le\ell\le k+1$ and $\lambda^2+(n-\ell+\frac{3}{2})^2$ when $k+2\le\ell\le n+1$. The numerators of these rational functions are given by

$$\begin{aligned}
2i\lambda\frac{P^n_{k,\ell}(\lambda)}{\Pi(\rho_M)}\Big|_{\lambda=i(n-k+\frac{1}{2})} &= (-1)^{k-\ell-1}2(n-k+\frac{1}{2})d(\sigma_\ell) \qquad \text{for} \quad 2\le\ell\le k+1,\\
i\lambda\frac{P^n_{k,\ell}(\lambda)}{\Pi(\rho_M)}\Big|_{\lambda=i(n-\ell+\frac{5}{2})} &= (n-\ell+\frac{5}{2})d(\sigma_k) \qquad \text{for} \quad 2\le\ell\le k+1,\\
i\lambda\frac{P^n_{k,\ell}(\lambda)}{\Pi(\rho_M)}\Big|_{\lambda=i(n-\ell+\frac{3}{2})} &= (n-\ell+\frac{3}{2})d(\sigma_k) \qquad \text{for} \quad k+2\le\ell\le n+1,
\end{aligned}$$

so that the sum of these rational functions over $2\le\ell\le n+1$ is

$$(-1)^k 2\sum_{1\le\ell\le k}(-1)^\ell d(\sigma_{\ell-1})\,\frac{(n-k+\frac{1}{2})}{\lambda^2+(n-k+\frac{1}{2})^2}+d(\sigma_k)\sum_{\substack{2\le\ell\le n+1\\ \ell\ne k+1}}\frac{(n-\ell+\frac{3}{2})}{\lambda^2+(n-\ell+\frac{3}{2})^2}. \tag{A.4}$$

Finally taking the terms in (A.1) and (A.4) with a polynomial denoted by $P_k(\lambda)$, we obtain

$$\begin{aligned}
\Omega(\sigma_k,\lambda) = &-\frac{d(\sigma_k)}{2}\Big(\psi(i\lambda-n-\frac{1}{2})+\psi(-i\lambda-n-\frac{1}{2})+\psi(i\lambda+1)+\psi(-i\lambda+1)\Big)\\
&+\frac{(-1)^k(n-k+\frac{1}{2})}{\lambda^2+(n-k+\frac{1}{2})^2}\left(\sum_{j=0}^{k-1}(-1)^j d(\sigma_j)+\sum_{j=k+1}^{2n+1}(-1)^{j+1}d(\sigma_j)\right)-P_k(\lambda).
\end{aligned}$$

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